MATH 468 - TEST 1, March 7,2001

Last Name(please print)

First Name

1. (6pt) Show that a compact metric space (X, d) is separable, i.e. there exists a countable subset $S \subset X$ which is dense in X, $\bar{S} = X$. (You know at least one example of separable space. The real line $\mathbb{R}$ is separable because the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$.)

2. (8pt) Consider the following two subsets of $\mathbb{R}$

$$A = \{0\} \cup \left\{ \frac{1}{n};\ \ n \in \mathbb{Z},\ \ n > 0 \right\},\ \ B := \left\{ n;\ \ n \in \mathbb{Z}, n \geq 0 \right\}$$

equipped with the subspace topologies.
(a) **(2pt)** Describe the open subsets in A and B.
(b) **(2 pt)** Show that the set $A - \{0\}$ equipped with the subspace topology is homeomorphic to B.
(c) **(2 pt)** Show that both A and B are closed.
(d) **(2 pt)** Show that A is **not** homeomorphic to B.

3. (4pt) Consider the sequence $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = n \sin(\frac{x}{n})$, $\forall x \in \mathbb{R}$, $n > 0$. Define $f_\infty : \mathbb{R} \to \mathbb{R}$ by

$$f_\infty(x) = x,\ \ \forall x \in \mathbb{R}.$$

(a)**(2pt)** Show that

$$\lim_{n \to \infty} f_n(x) = f_\infty(x),\ \ \forall x \in \mathbb{R}.$$

Hint: You need to know the fundamental result

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

(b) **(2pt)** Show that the sequence f_n does not converge uniformly to f_∞.

4. (6pt) Suppose (X, d) is a complete metric space and $F : X \to X$ is a function with the property that

$$d(F(x_1), F(x_2)) \leq \frac{1}{2} d(x_1, x_2).$$

Fix $x_0 \in X$ and define inductively

$$x_1 = F(x_0), \quad x_2 = F(x_1), \quad x_3 = F(x_2), \cdots, x_n = F(x_{n-1}), \quad x_{n+1} = F(x_n), \cdots.$$

(a) **(1pt)** Show that

$$\frac{1}{2} d(x_0, x_1) > d(x_1, x_2), \quad \frac{1}{4} d(x_0, x_1) > d(x_2, x_3), \cdots, \frac{1}{2^n} d(x_0, x_1) > d(x_n, x_{n+1}).$$

(b) **(1pt)** Show that

$$\frac{1}{2^k} d(x_n, x_{n+1}) > d(x_{n+k}, x_{n+k+1}), \quad \forall n, k > 0.$$

(c) **(2pt)** Use part (a) and (b) to prove that (x_n) is a Cauchy sequence.
(d) **(2pt)** Since (X, d) is complete, the Cauchy sequence (x_n) has a limit which we denote by x_∞. Show that $F(x_\infty) = x_\infty$, and

$$d(x_\infty, x_0) \leq 2d(x_1, x_0).$$

Note A point $x \in X$ such that $F(x) = x$ is called a **fixed point** of F. Problem **4** describes one method of detecting fixed points.