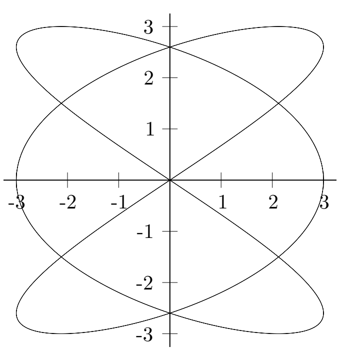
The goal of these notes is to describe some results in knot theory. Recall that a knot is a smooth embedding $e\colon S^1 \to \mathbf{R}^3$. One rarely writes an explicit formula for e, preferring instead to draw a *regular projection*, defined below. A projection for $\mathbf{R}^3$ is any linear map $p\colon \mathbf{R}^3 \to \mathbf{R}^2$ which is onto, or equivalently has a 1–dimensional kernel. The projection for the knot is the composition $p \circ e$.

The knot projection is an *immersion* provided the differentiable of $p \circ e$ is non–zero at each point in S^1. To actually do calculations, think of e as a function $\mathbf{R}^1 \to \mathbf{R}^3$ which is periodic and then we can use multi–variable calculus as in 225. The differential of e at a point $x \in S^1$ is a linear map $de_x\colon \mathbf{R}^1 \to \mathbf{R}^3$ which is 1 to 1 since e is an embedding (local embedding will suffice for this result). Since the kernel of p is 1 dimensional, usually $d(p \circ e)_x$, which is a linear map from $\mathbf{R}^1$ to $\mathbf{R}^2$, has a 0 dimensional kernel.

Any immersion $f\colon S^1 \to \mathbf{R}^2$ is locally 1 to 1. This is a corollary of the Implicit Function Theorem, but can be proved in this case as follows. Pick any point $x \in S^1$. Then the differential of $p \circ e$ evaluated at x is a non–zero vector in $\mathbf{R}^2$, say (a, b). Let $q\colon \mathbf{R}^2 \to \mathbf{R}^1$ be the projection $q(x, y) = ax + by$. The composition $q \circ p \circ e$ is a map $S^1 \to \mathbf{R}^1$ with derivative at x equal to $a^2 + b^2 \neq 0$. By the Inverse Function Theorem, the composite is 1 to 1 in a small neighborhood of x and hence so must be $p \circ e$. Since $p \circ e$ is locally 1 to 1 and since S^1 is compact, the inverse image of any point in $\mathbf{R}^2$ is a finite set. A point in $\mathbf{R}^2$ whose inverse image consists of 1 point is called a *regular point*; a point whose inverse image consists of 2 points is called a *double point*. A double point is called *isolated* if there exists a neighborhood in which all the other points are regular. An isolated double point is called *transverse* provided the two tangent vectors at the double point are linearly independent, which in this case means not parallel. A *regular immersion* is an immersion which has only transverse double points and regular points.

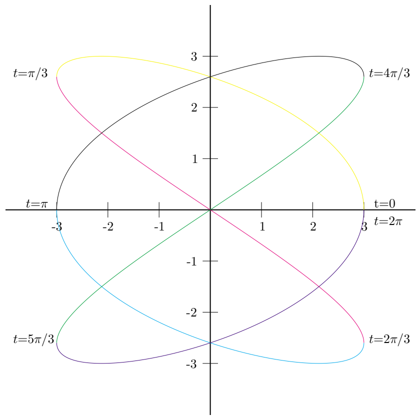
Here is an example: the function $e(t) = (3\cos 3t, 3\sin 2t, \sin t)$ is a smooth function from $\mathbf{R}^1$ to $\mathbf{R}^3$ which is periodic with period 2π and so is a function $S^1 \to \mathbf{R}^3$. We let the projection $p\colon \mathbf{R}^3 \to \mathbf{R}^2$ be the linear function $p(x, y, z) = (x, y)$. The graph of $p \circ e(t) = (3\cos 3t, 3\sin 2t)$, which is an example of a Lissajous figure, is



which looks like a regular immersion. It actually will be a regular immersion provided that it is an immersion and that the only intersection are the evident transverse ones.

The differential of $p \circ e$ is $(-9\sin 3t, 3\cos 2t)$ and this vector is never 0, so $p \circ e$ is an immersion.

To calculate the intersection points we need to find all solutions to the equations $(3\cos 3t, 3\sin 2t) = (3\cos 3s, 3\sin 2s)$ with $0 \leq t \neq s < 2\pi$. With no help, this can be tricky, but we can use the graph and the derivative. There are apparently 6 points where a particle moving along our curve changes direction with respect to its motion along the x–axis. From 0 to $\frac{\pi}{3}$, $\sin 3t \geq 0$ so $-9\sin 3t \leq 0$ so the particle moves to the left. Since $-9\sin 3t$ is only 0 at the end points, the motion is uniform. From $\frac{\pi}{3}$ to $\frac{2\pi}{3}$ $-9\sin 3t \geq 0$ so the particle moves steadily to the right. Continuing in this way we see the particle travels the curve as indicated by the next graph. Each colored arc is an embedding because the particle never reverse direction along the x–axis on an arc of a fixed color.



The intersection points can now be worked out.

1) $(\frac{3\sqrt{2}}{2}, \frac{3}{2})$: $t = \frac{\pi}{12}$ on the yellow curve; $t = \frac{17\pi}{12}$ on the green curve.
2) $(0, \frac{3\sqrt{3}}{2})$: $t = \frac{\pi}{6}$ on the yellow curve; $t = \frac{7\pi}{6}$ on the black curve.
3) $(-\frac{3\sqrt{2}}{2}, \frac{3}{2})$: $t = \frac{5\pi}{12}$ on the red curve; $t = \frac{13\pi}{12}$ on the black curve.
4) $(0, 0)$: $t = \frac{\pi}{2}$ on the red curve; $t = \frac{3\pi}{2}$ on the green curve.
5) $(-\frac{3\sqrt{2}}{2}, -\frac{3}{2})$: $t = \frac{7\pi}{12}$ on the red curve; $t = \frac{23\pi}{12}$ on the purple curve.
6) $(0, -\frac{3\sqrt{3}}{2})$: $t = \frac{11\pi}{6}$ on the purple curve; $t = \frac{5\pi}{6}$ on the blue curve.
7) $(-\frac{3\sqrt{2}}{2}, -\frac{3}{2})$: $t = \frac{19\pi}{12}$ on the blue curve; $t = \frac{11\pi}{12}$ on the green curve.

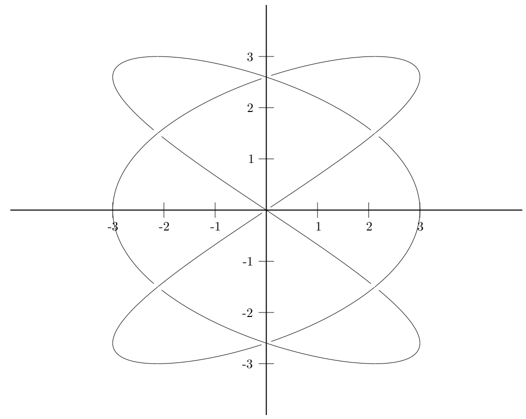
The solutions were found by eye–balling the intersections and then verifying the needed equalities. One can prove that these are the only intersections and that the intersections are transverse, although this is clear from the graph. Since the z–coordinate of e is $\sin t$, the z–coordinate is positive between 0 and π and negative between π and 2π. It is easy to see that e is an embedding, so e is a knot and this is a regular projection of it.

It is however an uninteresting knot since it is unknotted. Why?

A more interesting function is $e(t) = (3\cos 3t, 3\sin 2t, \sin 3t)$. It has the same projection as our first example since the first two coordinates are the same. Hence the colored arcs alternate coming out of the plane at you (positive z–coordinate) and going into the plane (negative z–coordinate). This shows that the only possible intersections are the four points in the interiors of the quadrants (1, 3, 5 and 7 above). Unfortunately, the e has intersections at these values of t even in $\mathbf{R}^3$, so e is not an embedding.

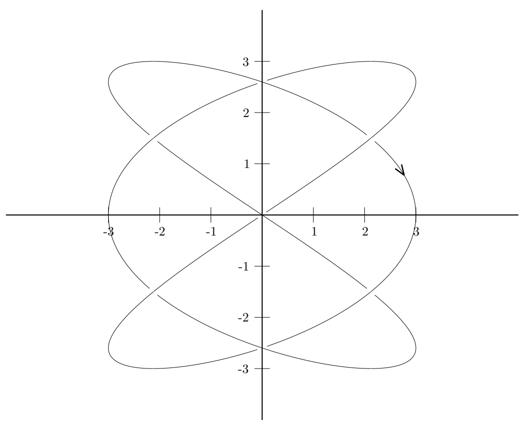
Consider $e(t) = (3\cos 3t, 3\sin 2t, 4\sin^2 t \sin 3t + 12\sin^{257} t)$. The xy–projection is still the same, so the only intersections this e could have are at the intersections of the projection. One can verify that here there are no intersections. We get a knot for which the xy–projection is regular:

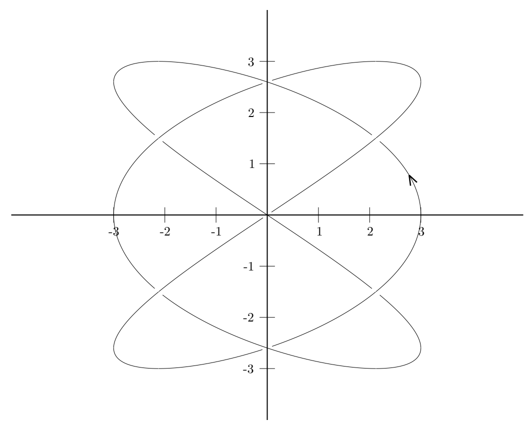
Note that this knot is *alternating*, (which means that the crossings alternates between over and under). It is a fact that an alternating knot (or link) with more than one crossing is non–trivial.



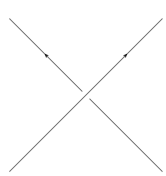
A knot is an embedding $e\colon S^1 \to \mathbf{R}^3$ and so will have some parameterization. We can think of the parameterization as a particle moving along the curve by the formula $e(t)$. The parameterization gives an *orientation* for the knot since an orientation is just a direction to travel around the knot. Since the derivative of the embedding is always non–zero, a particle moving along the knot never stops and hence never reverses direction.

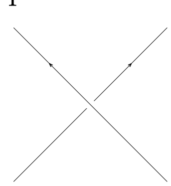
Since $p \circ e$ is an immersion for a regular projection of the knot, we also see that the parameterization of the projection determines an orientation. In practice, we draw a regular projection with no formulas for the knot (or link or even an embedded graph). To orient a knot, we just draw an arrow on the regular projection. For the example above we have the two pictures:



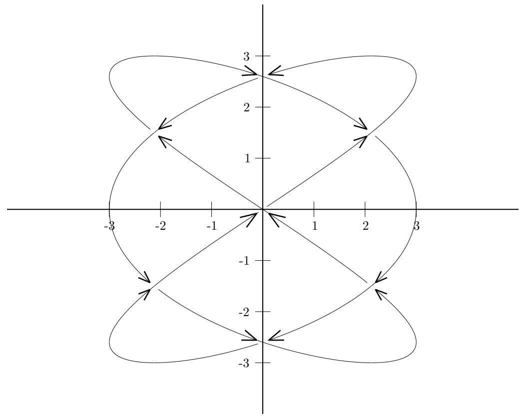


In a regular knot, the double points are isolated, so we can isolate each intersection. That is we can find a small disc centered at the double point and within that disk all we see is two stands of the knot with one crossing. An orientation on a knot allows us to assign a handedness to the double point.

If the crossing looks like  we call it a *right hand crossing*.

A precise description of a right hand crossing follows. Place your right hand with fingers straight along the over crossing with your fingers pointing in the direction of the orientation arrow and with your thumb pointing up. When you curl your fingers in the natural direction to lie along the under crossing, your fingers should point in the direction of the arrow on the under crossing. A *left hand crossing* looks like  .

In the example above, we can orient the knot and then check that all the crossings are left handed ones.



Reversing the orientation does not change the type of the crossing, a point which is easily checked in this example. Reversing a crossing does change the type of a crossing from right to left and vice versa.

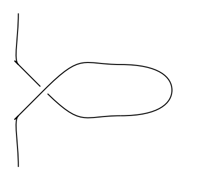
We can associated an integer invariant, called the *writhe*, to a regular projection. The writhe is just the number of right handed crossing minus the number of left handed crossing. In our example, the writhe is -7.

Reidemeister Moves.

One can give algorithms for associating numbers or groups to regular projections, but we want to get invariants of the knot, not just the regular projection.

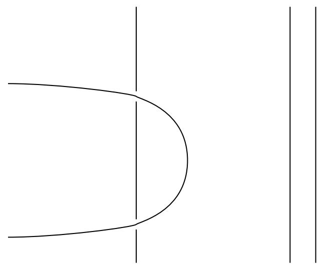
Reidemeister solved this problem in the 1920's by proving that one can get from one regular projection of a knot to another regular projection for the same knot by a sequence of three kinds of moves, now called the Reidemeister moves.

First are the *type I* Reidemeister moves:



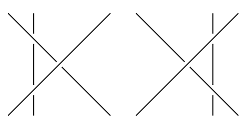
The meaning of the illustration is that we are looking at a piece of the knot projection where we see either the strand on the left (the one with the kink) or the one on the right (the straight line). Whichever picture we see, we can get a new picture by replacing the one we see by the other. There is a second type I Reidemeister move: the picture is the same except that the kink has the other crossing. In neither case is it necessary that the picture be vertical as shown. Type I Reidemeister moves just say that you may add a kink any where you like as long as it doesn't interfere with the other parts of the picture. You may also remove a kink anytime you see one.

Type II Reidemeister moves are illustrated by the following picture.



Type II Reidemeister moves allow you to replace a pair of strands which do not cross by a picture with two double points. They also allow you to remove a pair of double points provided the crossing strand is either above the other or below it. The illustration has the bent strand on top, but there is another type II move where it is on the bottom.

Type III Reidemeister moves are illustrated by the following picture.



Here the vertical strand behind a crossing is passed to the other side of the crossing, all the while staying behind the two crossing strands. There are three other pictures: the strand stays in front; the crossing is changed and the strand is behind; the crossing is changed and the strand stays in front.

One is rarely interested in actually producing the sequence of Reidemeister moves to get from one regular projection to another, but it is a very useful result for producing invariants. One need only check that your invariant doesn't change under the three types of moves.

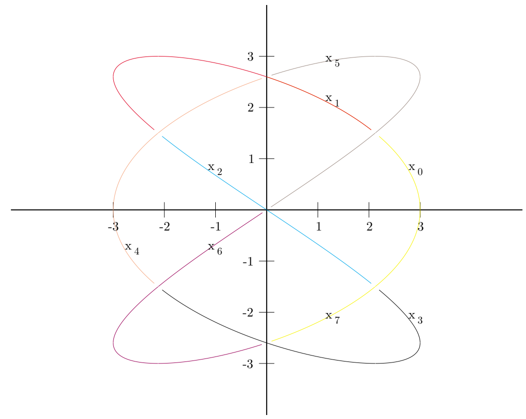
There is no issue of orientations here. Suppose given two regular projections of the same oriented knot. Both regular projections are oriented. Construct a sequence of Reidemeister moves between the two projections. Note that there is no doubt as to how to orient each stage of this sequence and that these orientations are the ones induced from the fixed orientation of the knot.

It is a worthwhile exercise to check that the writhe is invariant under type II and type III Reidemeister moves, but it is changed by a type I move. Hence the writhe is not an invariant of the knot. Moreover, given two regular projections of the same knot with different writhes, you know that you will have to use type I Reidemeister moves to get from one to the other.

Colorings.

To *color* a knot one first must locate the *arcs*. These are the arcs which begin at one under crossing and continue until the next and they are just connected arcs in the regular projection. Note that the number of these arcs is equal to the number of crossings. Here is our usual example with the arcs indicated by different colors.

To *color* the knot is to pick a prime p and then assign an integer to each arc so that at each crossing the equation $u_1 + u_2 = 2v \bmod p$ is satisfied, where v is the integer associated to the over crossing and u_1 and u_2 are the two integers associated to the two under crossings.



Given two regular projections for a knot which differ by a Reidemeister move, check that one has a coloring mod p if and only if the other does. Hence the existence of a mod p coloring is an invariant of the knot.

To see whether one can find a p and a set of integers to associate to the arcs is a non–trivial problem. These are linear equations so we can work in vector spaces over the field with p elements.

Let us work out the theory for our example. Let x_1 be the integer associated to the yellow arc and let x_2 be the integer associated to the red arc. Continue around the knot in the direction we've started to get $x_3, \ldots, x_7$. Number the crossings: 1 labels the crossing where the under crossing changes from yellow to red; 2 is the next crossing travelling in this direction; 3 the next and so on. The equations in this basis are described by a 7×7 matrix. Begin with the matrix

$$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

and add one -2 to each row depending on the over crossing.

$$\begin{matrix} 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -2 & 0 & 0 & 1 \end{matrix}$$

The knot has a mod p coloring if and only if the subspace of solutions to this 7×7 system of homogeneous equations has dimension at least 2. By construction, the columns add up to the zero vector so the solution space always has dimension at least one.

The solution space to our system has dimension greater than 1 if and only if

$$\det\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 & -2 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 1 \end{pmatrix}$$

is divisible by p, where the displayed matrix is the 6×6 matrix obtained by dropping the last row and column. But this determinant is 15, so there are solutions mod 3 and mod 5. Work out actual solutions if you wish.

It is also fun to observe that no matter which knot you have, the determinant is always odd. This is no longer true for a link.

Groups.

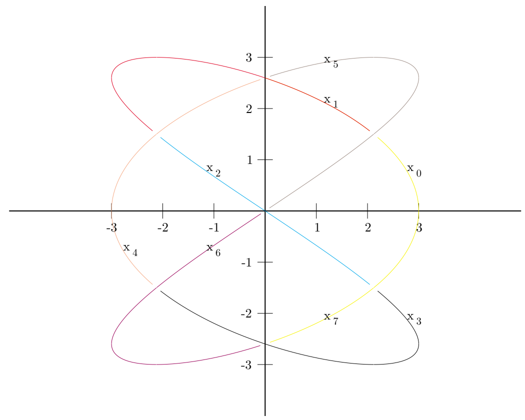
One can also associate a group to a knot as follows. Orient the knot and fix a group G. Assign to each arc an element of the group $x_i \in G$ so that the following equations are satisfied. Suppose the two under crossings are assigned elements u_1 and u_2 with u_1 coming before u_2 as you follow the knot in the direction indicated by the orientation. Let v be the element associated to the over crossing. If the crossing is right handed, then the equation is $u_2 = vu_1v^{-1}$; if the crossing is left handed, then the equation is $u_2 = v^{-1}u_1v$. The elements generate a subgroup of G and we associate this subgroup to the knot. One can check that the associated group only depends on the regular projection up to Reidemeister moves and so is an invariant of the knot.

One theoretical result is to notice that if $x \in G$ is the element assigned to the first arc, then the elements associated to the other arcs are conjugates of x. We can also replace x by any element in its conjugacy class and alter the other assignments appropriately to get a new solution. Hence we have not only assigned a group to the knot, but also a conjugacy class of elements within the group. The elements of this conjugacy class must generate the group. In general, not every element in a conjugacy class will be assigned to an arc, but by using type II Reidemeister moves, one can find a new regular projection in which every element in the conjugacy class is assigned to at least one segment.

There is a universal group in which our equations are satisfied. If you have not studied these ideas, this paragraph can be skipped. To continue, the universal group associated to a regular projection is given by generators and relations. The generators are the arcs and there is one relation from each crossing. Denote this group by G_P where P is the oriented projection. All of the x_i are conjugate in this group and this is the associated conjugacy class. This group is universal in the following sense. Given any group G and elements $g_i \in G$, there exists a (necessarily unique) homomorphism $h: G_P \to G$ such that $h(x_i) = g_i$ if and only if the g_i satisfy the same relations as the x_i. As an example, if we are given a group G associated to the knot, then there is a unique homomorphism $\varphi: G_P \to G$ which is onto and which takes the x_i assigned to an arc to the $g_i \in G$ assigned to the same arc. It is worth remarking that if we make G_P abelian by dividing out by the commutator subgroup, the quotient group is $\mathbf{Z}$ since in the quotient all the x_i are equal. It further follows that all the x_i map either to $1 \in \mathbf{Z}$ or they all map to -1. Moreover, any group associated to a knot must have a cyclic abelianization. Groups which have this property include dihedral groups, symmetric groups, simple groups and cyclic groups.

There is one more bit of structure to be squeezed out of our setup. Pick one arc on which to begin. Then follow the knot in the direction of the orientation. Construct an element of your group as a product. Each time you cross under an arc, write down the element associated to the over crossing if the crossing is *left* handed, otherwise write down the inverse of that element. When you have gone once around the knot, append the element assigned to your initial segment raised to the writhe of the regular projection. Call the result ℓ_P: it is called a *longitude*. The conjugacy class of ℓ_P is a knot invariant. We discuss the proof below.

For P the regular projection we have been studying, G_P is generated by $x_0, \ldots x_6$ with relations $x_1 = x_5^{-1} x_0\, x_5$, $x_2 = x_4^{-1} x_1\, x_4$, $x_3 = x_0^{-1} x_2\, x_0$, $x_4 = x_6^{-1} x_3\, x_6$, $x_5 = x_1^{-1} x_4\, x_1$, $x_6 = x_2^{-1} x_5\, x_2$ and $x_0 = x_4^{-1} x_6\, x_4$. All the crossings are left handed so the writhe is -7. A longitude is $\ell_P^{(0)} = x_5 x_4 x_0 x_6 x_1 x_2 x_3 x_0^{-7}$. If, instead of starting at x_0, which is a choice after all, we start at x_1, we get $\ell_P^{(1)} = x_4 x_0 x_6 x_1 x_2 x_3 x_5 x_1^{-7}$. Check $\ell_P^{(0)} = x_5 \ell_P^{(1)} x_5^{-1}$.



If we calculate $x_0 \ell_P^{(0)}$, we get $\ell_P^{(0)} x_0$. Check this for the example above. For the general case, consider the relation between the group G_P and the longitude. Label the arcs x_0, $\ldots$, x_{r-1} where x_0 is a chosen starting arc and then the remaining ones are encountered in order as one travels around the knot in the given direction. As one goes around the knot in the preferred direction, one encounters over crossings at the end of each x_i: let x_{j_i} be the label for the over crossing. If the writhe of the projection is w, the longitude is $\ell_P^{(0)} = x_{j_0}^{\epsilon_0} x_{j_1}^{\epsilon_1} \cdots x_{j_{r-1}}^{\epsilon_{r-1}} x_0^w$, where $\epsilon_i = +1$ if the crossing at then end of x_i is a left hand crossing and -1 if it is right handed. The relations are $x_{i+1} = x_{j_i}^{-\epsilon_i} x_i x_{j_i}^{\epsilon_i}$ for $0 \le i < r-1$ and $x_0 = x_{j_{r-1}}^{-\epsilon_{r-1}} x_{r-1} x_{j_{r-1}}^{\epsilon_{r-1}}$. If we agree to write subscripts mod r, then this last relation can be written as $x_{(r-1)+1} = x_{j_{r-1}}^{-\epsilon_{r-1}} x_{r-1} x_{j_{r-1}}^{\epsilon_{r-1}}$ so all the relations have the same form. These relations can be rewritten as $x_{j_i}^{\epsilon_i} x_{i+1} = x_i x_{j_i}^{\epsilon_i}$ for $1 \le i < r$. It is now easy to calculate $x_0 \ell_P^{(10)} = x_{j_0}^{\epsilon_1 0} \cdots x_{j_{i-1}}^{\epsilon_{i-1}} x_i x_{j_i}^{\epsilon_i} \cdots x_{j_{r-1}}^{\epsilon_{r-1}} x_0^w$ for each $1 \le i < r-1$ and then finally $x_0 \ell_P^{(0)} = x_{j_0}^{\epsilon_0} x_{j_1}^{\epsilon_1} \cdots x_{j_{r-1}}^{\epsilon_{r-1}} x_0 x_0^w = \ell_P^{(0)} x_0$. In other words, x_0 and $\ell_P^{(0)}$ commute.

If we start with x_i, we get longitude $\ell_P^{(i)} = x_{j_i}^{\epsilon_i} x_{j_{i+1}}^{\epsilon_{i+1}} \cdots x_{j_{r-1}}^{\epsilon_{r-1}} x_{j_0}^{\epsilon_0} \cdots x_{j_{i-1}}^{\epsilon_{i-1}} x_i^w$, and we see that x_i and $\ell_P^{(i)}$ commute. Moreover $\ell_P^{(i+1)} = x_{j_i}^{-\epsilon_i} \ell_P^{(i)} x_{j_i}^{\epsilon_i}$ and we have $x_{i+1} = x_{j_i}^{-\epsilon_i} x_i x_{j_i}^{\epsilon_i}$, so the pairs $\left\{x_i, \ell_P^{(i)}\right\}$ are all conjugate. The element x_i is called a *meridian* for the longitude $\ell_P^{(i)}$ and the pair $\left\{x_i, \ell_P^{(i)}\right\}$ is called a *meridian–longitude pair*. Recall that in the abelianization of G_P, x_i went to ± 1. Any longitude goes to 0 in the abelianization since

$\sum_{i=1}^{r} \epsilon_i = -w$. It follows that it is not necessary to stress which element in a meridian–longitude pair is the meridian and which is the longitude.

One can also check that the conjugacy class of a meridian–longitude pair is invariant under the Reidemeister moves. A deep theorem of Waldhausen from the 1960's says that two knots are equivalent if and only if there is an isomorphism $h\colon G_P \to G_{P'}$ such that $\{h(x_1), h(\ell_P^{(1)})\}$ is a meridian–longitude pair for P'. Again, we do not need to worry over who is the meridian and who is the longitude since no isomorphism can interchange them. There do exist examples of distinct knots with the same group, that is there is an isomorphism between the groups, but none that takes a meridian–longitude pair for the first knot to a meridian–longitude pair for the second knot.

If we reverse the orientation on the knot, we have the same arcs and over crossings and handedness of each crossing, but the algebra looks a little different since both the relations and the longitude depend on the direction we travel along the knot. Let x_i, ϵ_i, j_i be as above. Start with the same arc to which we assigned x_0 and let y_0 be the corresponding generator. Then travel around the knot in our new preferred direction to get generators $y_1, \ldots, y_{r-1}$. Note the same arc is labelled x_i going one way and y_{r-i} going the other. Let k_i be the subscripts for the over crossings and δ_i the sign. The crossing labelled i in the current picture is labeled $r-1-i$ in the old picture, and since the handedness is independent of the orientation, $\delta_i = \epsilon_{r-1-i}$. Also $k_i = r - j_{r-1-i}$. The relations are $y_{i+1} = y_{k_i}^{\delta_i} y_i y_{k_i}^{-\delta_i}$, $0 \leq i < r$. The function $y_i \mapsto x_{r-i}^{-1}$ extends to an isomorphism $\iota\colon G_{P^-} \to G_P$, where P^- denotes P with the opposite orientation. To check this we need to see $x_{r-(i+1)}^{-1} = x_{j_{r-1-i}}^{-\epsilon_{r-1-i}} x_{r-i}^{-1} x_{j_{r-1-i}}^{\epsilon_{r-1-i}}$. But this relation is equivalent to $x_{r-i} = x_{j_{r-i-1}}^{\epsilon_{r-i-1}} x_{r-i-1} x_{j_{r-i-1}}^{-\epsilon_{r-i-1}}$ and this is one of the relations in G_P. A longitude is $\ell_{P^-}^{(0)} = y_{k_0}^{\delta_0} \cdots y_{k_{r-1}}^{\delta_{r-1}} y_0^w$. Check $\iota(\ell_{P^-}^{(0)}) = x_{j_{r-1}}^{-\epsilon_{r-1}} \cdots x_{j_0}^{-\epsilon_0} x_0^{-w}$. Since x_0 and $x_{j_0}^{\epsilon_0} x_{j_1}^{\epsilon_1} \cdots x_{j_{r-1}}^{\epsilon_{r-1}}$ commute, $\iota(\ell_{P^-}^{(0)}) = (\ell_P^{(0)})^{-1}$. Summarizing, we see that G_P is the group for the knot with the opposite orientation but when we write the group this way, $\{x_0^{-1}, (\ell_P^{(0)})^{-1}\}$ is a meridian–longitude pair for the knot with the opposite orientation.

It is hard to prove but there are oriented knots so that no deformation of three space throws the knot with its initial orientation onto the same picture but with the opposite orientation. Such knots are called *non–invertible*. The first such knots were not identified until 1967 by Trotter. If an oriented knot can be deformed through ambient isotopies to the same picture but with the opposite orientation, the knot is called *invertible*.

Theorem: A knot is invertible if and only if there is an automorphism $\iota\colon G_P \to G_P$ such that $\iota(x_0) = x_0^{-1}$ and $\iota(\ell_P^{(0)}) = (\ell_P^{(0)})^{-1}$.

Recall that an *automorphism* of a group is another name for an isomorphism from the group to itself.

Groups can be used to study the chiral problem, but not obviously since when all the crossing are switched in a diagram, the arcs change in a manner that depends on the knot. Using the description of the group of the knot as the fundamental group of $\mathbf{R}^3$ minus the knot, one can prove the result below. First observe that there are two types of achirality. A knot can be $(+)$–achiral or $(-)$–achiral: $(+)$–achiral means that there is an ambient isotopy of 3–space which sends the knot onto its mirror image but with the same orientation; $(-)$–achiral means that there is an ambient isotopy of 3–space that sends the knot onto its mirror image but with the opposite orientation. The only if part of the next result follows from Waldhausen's theorem.

Theorem: A knot is $(+)$–achiral if and only if there is an automorphism $\alpha_+\colon G_P \to G_P$ such that $\alpha_+(x_0) = x_0^{-1}$ and $\alpha_+\big(\ell_P^{(0)}\big) = \ell_P^{(0)}$. A knot is $(-)$–achiral if and only if there is an automorphism $\alpha_-\colon G_P \to G_P$ such that $\alpha_-(x_0) = x_0$ and $\alpha_-\big(\ell_P^{(0)}\big) = \big(\ell_P^{(0)}\big)^{-1}$.

An older name for achiral is *amphicheiral*.

While these two theorems are correct, they are of limited usefulness since it is usually difficult to prove the non–existence of isomorphisms with prescribed properties.

Exercise: The left–handed trefoil group is generated by two elements, subject to one relation $G = \big\langle x_0, x_2 \mid x_0x_2x_0 = x_2x_0x_2 \big\rangle$, where both x_0 and x_2 are meridians. A longitude commuting with x_0 is $\ell^{(0)} = x_2x_0^2x_2x_0^{-4}$. Show the left hand trefoil is invertible by producing an automorphism of G which takes x_0 to x_0^{-1} and $\ell^{(0)}$ to $\big(\ell^{(0)}\big)^{-1}$.