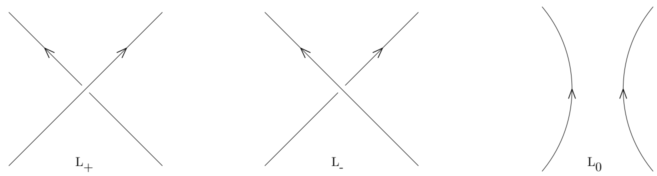
We turn to some results on the HOMFLY polynomial. Recall that this is a Laurent polynomial in two variables, traditionally ℓ and m, satisfying three conditions.

1) $P(L)$ is an invariant of the oriented link L, not just the particular projection given.
2) $P(\text{unknot}) = 1$.
3) If L_+, L_- and L_0 are related as below,

$$(*) \qquad \ell P(L_+) + \ell^{-1} P(L_-) + m P(L_0) = 0$$



Next we record some results for later. The number of components of L_+ and L_- are the same. The number of components of L_0 changes. If the two strands under discussion belong to the same component of the link L_+ (which is equivalent to belonging to the same component of L_-) then L_0 has one more component; if the two strands belong to different components, then L_0 has one less component than L_+ and L_-.

The next order of business is an algorithm for computing $P(L)$. In practice, this is not always the most efficient method for doing computations, but it is invaluable for proving theorems.

Theorem 1. *If $L \amalg S^1$ denotes the link formed from a link L by adding an unknotted circle inside a ball which misses L, $P(L \amalg S^1) = -(\ell + \ell^{-1}) m^{-1} P(L)$.*

Corllary 2. *If L is the unlink on $n+1$ components, $P(L) = (-1)^n (\ell + \ell^{-1})^n m^{-n}$.*

Proof: The proof of the corollary is immediate from the theorem. To prove the theorem, consider the effect of a type I Reidemeister move on a small straight arc of L, going from a straight segment to one with one kink. Switching the crossing switches the handedness of the kink, so both L_+ and L_- are just L. The link L_0 is just $L \amalg S^1$. Solving the equation $(*)$ yields the formula. ∎

A result to be proved later is

Theorem 3. *By changing some of the crossings in a regular projection, any regular projection can be changed into a regular projection for the unlink.*

Remark: In fact, we will give an algorithm for doing this.

The theorem says that given a link, we can mark some of the crossings to be switched, say r of them, so that, when we have switched all r of them, we have the unlink. Consider a crossing to be switched. Let ϵ be $+1$ if the crossing is a right handed one and -1 if it is left handed. Let L' be the link obtained by switching the crossing and let L_0 be the link obtained by splitting the crossing. Then $(*)$ yields

$$P(L) = -\ell^{-2\epsilon} P(L') - \ell^{-\epsilon} m P(L_0) \ .$$

Now notice that if $\omega(\ )$ denotes the writhe, $2\epsilon = \omega(L) - \omega(L')$ and $\epsilon = \omega(L) - \omega(L_0)$, so we can rewrite the formula as

$$P(L) = -\ell^{\,\omega(L')-\omega(L)}P(L') - \ell^{\,\omega(L_0)-\omega(L)}mP(L_0)\ .$$

If we define $Q(L) = \ell^{\,\omega(L)}P(L)$ then we get

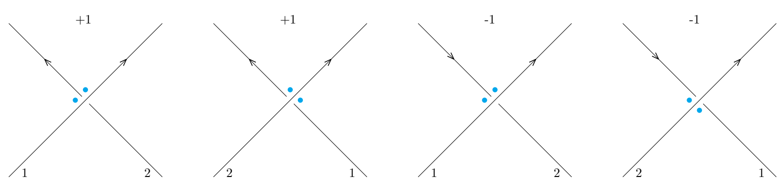
$$Q(L) = -Q(L') - mQ(L_0)\ , \tag{4}$$

a much simpler formula with which to work. The drawback is that Q is not an invariant of the link, only of the projection, but since the writhe is not so bad to calculate, working with a fixed projection is not too bad in practice. Actually, Q is a bit better than the last sentence says. Given two regular projections of the same link with the same writhe, the two Q's for these projections are the same. So if you redraw the projection to keep the link the same, just compute the before and after writhes and put some type 1 Reidemeister moves in your new projection to make the writhes the same and keep going. One last formula: if L is an unlink on $n+1$ components, then $Q(L) = (-1)^n(\ell + \ell^{-1})^n\ell^{\,\omega(L)}m^{-n}$.

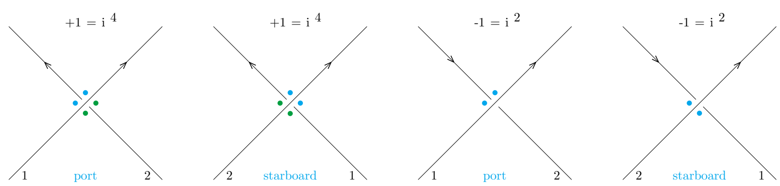
If one is going to be computing a lot of writhes, it is good to have an easy way of doing this. Here is an algorithm which computes the writhe in one trip around the oriented link. Order the components (the order doesn't matter) and pick a starting point on each component (which doesn't matter either). Given the close relations over the years between sailors and knots, it is a bit surprising that more nautical terminology hasn't crept into knot theory, but here goes. The *port side* of a strand on a regular projection of an oriented link is the left side as you face in the direction determined by the orientation. The *starboard side* is the right side as you face in this direction.

To compute the writhe, travel around the link in the direction of the orientation beginning at the chosen point on the first component. When you return to this point, move to the second component and repeat until you have been once around each component. Recall that you will go through each crossing twice. The first time through a crossing, draw two dots. If you are going along an over crossing, put two dots on the port side, one on each side of the under crossing. If you are on the under crossing, put two dots on the starboard side, one on each side of the over crossing. Whenever you encounter a crossing that already has two dots, add a -1 if the dots come *before* the crossing or add $+1$ if they come *after*. This sum computes the writhe. Indeed, the ± 1 you compute at each crossing is the sign of that crossing.

To see this consider the figure below. The orientations on the strands are indicated by the arrows. The numbers (1 or 2) at the bottom of each strand indicate the order in which they were traversed. (This of course depends on the rest of the link.) The blue dots are the dots and the sign of the crossing is indicated at the top. Convince yourself that the sign given by the algorithm is the same as the sign given by the handedness and that any oriented crossing with a given order on the strands can be rotated to one of these four pictures.



Here is a variation on the same algorithm. As before number the components and pick starting points. Traverse the link as in the first algorithm and the first time you come to a crossing, put down two dots as before. The second time you come to a crossing if the dots come after the crossing, put two more dots down on the near side of it. The sign of the crossing is just $i^{\text{ number of dots}}$ where $i = \sqrt{-1}$. The next picture shows this version of the algorithm, where we use green dots, when we need them, for the second pair.



Let us next calculate the HOMFLY polynomial for each trefoil. Let $\epsilon = +1$ for the right handed trefoil and -1 for the left. Let Tr_{+1} denote the right trefoil and Tr_{-1} the left. Both the right and left trefoils can be unknotted by switching a single crossing and the resulting unknot has writhe ϵ. In both cases, L_0 is a two component link with two crossings in the evident regular projection. They are both Hopf links with writhe 2ϵ. Denote them by Hp_ϵ. Switching either of the crossings transforms the link to the unlink on two components and the resulting split link is a one component link with one crossing, hence the unlink. The two component link has writhe 0 and the unknot has writhe ϵ. Using (4) gives $Q(Tr_\epsilon) = -\ell^\epsilon \cdot 1 - mQ(Hp_\epsilon)$ and $Q(Hp_\epsilon) = -\big(-(\ell+\ell^{-1})m^{-1}\big)^1 - m\ell^\epsilon \cdot 1 = (\ell + \ell^{-1})m^{-1} - m\ell^\epsilon$. Hence $Q(Tr_\epsilon) = -\ell^\epsilon + m^2\ell^\epsilon - (\ell + \ell^{-1}) = m^2\ell^\epsilon - 2\ell^\epsilon - \ell^{-\epsilon}$. Since the writhe of Tr_ϵ is 3ϵ, $P(Tr_\epsilon) = \ell^{-3\epsilon}Q(Tr_\epsilon) = m^2\ell^{-2\epsilon} - 2\ell^{-2\epsilon} - \ell^{-4\epsilon}$.

Errata: Looking at Figure 2.15 on page 47 and the answer for its HOMFLY polynomial given in the next to the last paragraph, Flapan has confused the left and right trefoil.

Notice that the trefoil is chiral since the mirror image of Tr_ϵ is $Tr_{-\epsilon}$ and the two polynomials are different. Recall that $\mathbf{Z}[\ell, \ell^{-1}, m, m^{-1}]$ is the free abelian group on all monomials of the form $\ell^r m^s$, r, $s \in \mathbf{Z}$. Since $P(Tr_{+1})$ and $P(Tr_{-1})$ have different coefficients on the monomial ℓ^{-4}, they are not equal.

Our first theorem is a negative one.

Theorem 5: *The HOMFLY polynomial for L is the same as for the link L with all the orientations reversed.*

Proof: Pay careful attention to the form of the proof as it will be repeated a great deal in the results to follow. We first introduce a certain amount of notation we will need in this and subsequent proofs.

The unlinking number of a link projection is the minimum number of crossings we need to switch to get a projection for the unlink. It exists by Theorem 3. If L is a regular link projection, let $u(L)$ denote the unlinking number. It can be difficult to compute in practice, but we know it exists. Now there are $u(L)$ crossings in L such that if we switch them all, we get an unlink. If we just switch one of them, we get a new link, L', and $u(L') = u(L) - 1$.

We denote the number of crossings in a regular link projection L by $c(L)$. If we take any crossing and form the resulting L_0, $c(L_0) = c(L) - 1$. These two formulas are the basis of many proofs by induction. Notice that if $u(L) = 0$ or $c(L) = 0$, then L is an unlink.

Let $\mathcal{S}$ be the set of all regular link projections for which the result holds.

Let $\bar{L}$ denote the result of reversing the orientation on *all* the components of L. We have $\omega(L) = \omega(\bar{L})$, so it suffices to prove $Q(L) = Q(\bar{L})$. If L is an unlink, $\bar{L} = L$ and so, any regular projection for the unlink has $Q(L) = Q(\bar{L})$ so $L \in \mathcal{S}$. *This is usually the first step in all the proofs to follow: check the result for all regular projections of the unlink.*

We prove all regular link projections are in $\mathcal{S}$ by a double induction on $c(L)$ and $u(L)$. The inductive step is the following. Fix $k > 0$ and $u > 0$. *Assume* every regular link projection with $c(L) < k$ is in $\mathcal{S}$ and further *assume* that every regular link projection with $c(L) = k$ and $u(L) < u$ is in $\mathcal{C}$. Then *prove* that every regular link projection with $c(L) = k$ and $u(L) = u$ is in $\mathcal{S}$.

If we can do this, all regular link projections are in $\mathcal{S}$. If not, pick a regular link projection which is not in $\mathcal{S}$ and which has the smallest $c(L)$ amongst all such L: let $k = c(L)$. Since $k = 0$ implies L is an unlink, $k > 0$. By our choice of k, any regular link projection with $c(L) < k$ is in $\mathcal{S}$. From all the regular link projections with $c(L) = k$ pick one with the smallest $u(L)$: let $u = u(L)$. Since $u = 0$ implies L is an unlink, $u > 0$. If L is any regular link projection with $c(L) = k$ and $u(L) < u$, then we know $L \in \mathcal{S}$. But then our inductive step shows $L \in \mathcal{S}$, contrary to assumption.

The proof of the inductive step is reduced to working with (4) as follows. Pick a crossing so that the switched link L' satisfies $u(L') = u - 1 < u$ and then $c(L_0) = k - 1 < k$. Hence both L' and L_0 are in $\mathcal{S}$ and we have to use (4) to deduce $L \in \mathcal{S}$.

The details for this theorem follow. It doesn't matter if we switch the crossing and then reverse the orientations or first reverse the orientations and then switch the crossing: i.e. $\overline{L'} = (\bar{L})'$. Likewise $\overline{L_0} = (\bar{L})_0$.

Since $L' \in \mathcal{S}$, $Q(L') = Q(\ \overline{L'}\ )$ and since $\overline{L'} = (\bar{L})'$, $Q(L') = Q\big((\bar{L})'\big)$. Likewise, $Q(L_0) = Q\big((\bar{L})_0\big)$. Apply (4) to L to get $Q(L) = -Q(L') - mQ(L_0)$ and to $\bar{L}$ to get $Q(\bar{L}) = -Q\big((\bar{L})'\big) - mQ\big((\bar{L})_0\big)$ so $Q(L) = Q(\bar{L})$ and so $L \in \mathcal{S}$. ∎

The HOMFLY polynomial is however much better at detecting chirality. We saw this for the trefoils and we can generalize appropriately.

Given any regular link projection L, let L^* denote the projection obtained by switching all the crossings: L^* is a regular projection for the mirror image of L. There is a ring automorphism of $\mathbf{Z}[\ell, \ell^{-1}, m, m^{-1}]$ which takes ℓ to ℓ^{-1}. We write this as bar: $\overline{\ell^a m^b}$ =

$\ell^{-a}m^b$. Check that given two polynomials p_1 and p_2, $\overline{p_1+p_2} = \overline{p_1}+\overline{p_2}$ and $\overline{p_1 \cdot p_2} = \overline{p_1}\cdot\overline{p_2}$. Since $\overline{0} = 0$ and $\overline{1} = 1$, bar is a ring homomorphism and since it is its own inverse, it is a ring automorphism. (There is also an automorphism which takes m to m^{-1} but we will have no need of it.)

Theorem 6: $\quad P(L^*) = \overline{P(L)}$.

Corollary 7: *If a link L is achiral, $P(L) = \overline{P(L)}$.*

Proof: The corollary follows from the theorem since if L is achiral, $L^* = L$. Note $\omega(L^*) = -\omega(L)$ so our theorem is equivalent to $Q(L^*) = \overline{Q(L)}$. Let $\mathcal{S}$ be the set of regular link projections L with $Q(L^*) = \overline{Q(L)}$.

Unlinks are achiral and we can check that for an unlink L, $P(L) = \overline{P(L)}$ and hence $Q(L^*) = \overline{Q(L)}$ for them, so regular projections of an unlink are in $\mathcal{S}$. Check that for any regular link projection, $c(L^*) = c(L)$ and $u(L^*) = u(L)$.

As usual, let L' be L with a crossing changed so $u(L') = u(L) - 1$ and let L_0 be the link projection with the crossing split. Then $(L^*)' = (L')^*$ and $(L^*)_0 = (L_0)^*$. As usual, (4) shows $Q(L^*) = \overline{Q(L)}$. ∎

Remark: The converse to Corollary 7 is false. The first example for a knot is 9_{42} which satisfies the conclusion of the corollary but is chiral anyway.

The next result gives some structure to $P(L)$ as a Laurent polynomial in m. Write $P(L) = \sum_{-\infty}^{\infty} p_i^L(\ell)m^i$, where $p^L(\ell)$ is a Laurent polynomial in $\mathbf{Z}[\ell,\ell^{-1}]$. We also need the function $\nu(L)$ which is the number of components of L minus 1.

Theorem 8: *If $l^r m^s$ occurs with a non–zero coefficient in $P(L)$, then $r \equiv \nu(L)$ and $s \equiv \nu(L)$ mod 2. In particular $p_s^L(\ell) = 0$ if $s \not\equiv \nu(L)$ mod 2.*

Proof: If L is an unlink, Corllary 2 says $P(L) = (-1)^{\nu(L)}\left(\ell+\ell^{-1}\right)^{\nu(L)}m^{-^{\nu(L)}}$ which expands to $P(L) = (-1)^{\nu(L)}\left(\sum_{i=0}^{\nu(L)} \binom{\nu(L)}{i} \ell^{\nu(L)-i}\ell^{-i}\right)m^{-\nu(L)}$ so the non–zeros terms are $\ell^{\nu(L)-2i}m^{-\nu(L)}$. The theorem follows for the unlink.

Since we are dealing with powers of ℓ as well as those of m, it is easier to work with the equation $P(L) = -\ell^{-2\epsilon}P(L') - m\ell^{-\epsilon}P(L_0)$ and we may assume the theorem holds for L' and L_0. If $\ell^s m^r$ occurs with non–zero coefficient in $P(L)$ then $\ell^{s+2\epsilon}m^r$ must occur with non–zero coefficient in $P(L')$ or else $\ell^{s+\epsilon}m^{r-1}$ must occur with non–zero coefficient in $P(L_0)$. But then the theorem says $s + 2\epsilon \equiv \nu(L')$ or else $s + \epsilon \equiv \nu(L_0)$ mod 2. But $\nu(L') = \nu(L)$ and $\nu(L_0) = \nu(L_0) \pm 1$ and $\epsilon = \pm 1$ so the result for s follows. A similar calculation proves the congruence for r. ∎

Remark: One way to rephrase Theorem 8 is that $(\ell m)^{\nu(L)} \cdot P(L)$ is a Laurent polynomial in ℓ^2 and m^2.

Theorem 9: *Let L be a regular projection of a link. Then*

1) *If $p_s^L \neq 0$, then $-\nu(L) \leq s \leq 2c(L) - \nu(L)$.*
2) *$p_{-\nu(L)}^L(1) \equiv 2^{\nu(L)}$ mod $2^{\nu(L)+1}$. In particular, $p_{-\nu(L)}^L(\ell) \neq 0$.*
3) *For $\nu(L) < r$, $\quad p_{-\nu(L)+2r}^L(\ell) = (\ell + \ell^{-1})^{\nu(L)-r}\ \tilde{p}_{-\nu(L)+2r}^L(\ell)$ where $\tilde{p}_{-\nu(L)+2r}^L(\ell) \in \mathbf{Z}[\ell, \ell^{-1}]$.*
4) *$P(L)(\ell, \ell + \ell^{-1}) = (-1)^{\nu(L)}$.*

Proof: Check that 1), 2), 3) and 4) hold for any regular projection of the unlink on any number of components. Note $Q(L)$ decomposes the same way as P and the pieces are $q_s^L(L) = \ell^{\,\omega(L)} p_s^L(\ell)$. Observe that the q_s^L satisfy 1), 2) and 3) if and only if the p_s^L do. We can rewrite (4) as

$$q_s^L(\ell) = -q_s^{L'}(\ell) - q_{s-1}^{L_0}(\ell)\ .$$

Note that L_0 has either $\nu(L) - 1$ or $\nu(L) + 1$ components. If the two strands of the crossing lie in different components, then $\nu(L_0) = \nu(L) - 1$: if the two strands lie in the same component then $\nu(L_0) = \nu(L) + 1$. We introduce a bit of notation to simplify writing our argument. Let L_0^+ be the result of splitting the crossing if the number of components increases by 1 and let L_0^- be the result otherwise. Hence $\nu(L_0^{\pm}) = \nu(L) \pm 1$. Furthermore $\nu(L') = \nu(L)$.

As usual $c(L') = c(L)$ and $c(L_0^{\pm}) = c(L) - 1$.

We need to prove that L satisfies our conditions under the assumption that L' and $L_0^{\pm}$ do. If $q_s^L \neq 0$, then either $q_s^{L'} \neq 0$ or $q_{s-1}^{L_0^{\pm}} \neq 0$.

If $q_s^{L'} \neq 0$, $-\nu(L') \leq s \leq 2c(L') - \nu(L')$ and $s \equiv \nu(L')$ mod 2. The values of c and ν evaluated at L' are the same as their values evaluated at L so 1) follows.

If $q_{s-1}^{L_0} \neq 0$, then

$$-\nu(L_0) \leq s - 1 \leq 2c(L_0) - \nu(L_0)\ .$$

For L_0^+ our formula becomes $-\nu(L) - 1 \leq s - 1 \leq 2c(L) - 2 - \nu(L) - 1 < 2c(L) - 1 - \nu(L)$ from which 1) follows. For L_0^-, our formula becomes $-\nu(L) - 1 < -\nu(L) + 1 \leq s - 1 \leq 2c(L) - 2 - \nu(L) + 1 = 2c(L) - \nu(L) - 1$ from which 1) follows again.

To evaluate $q_{-\nu(L)}^L$ apply $q_{-\nu(L)}^L(1) = -q_{-\nu(L)}^{L'}(1) - q_{-\nu(L)-1}^{L_0}(1)$. By hypothesis $q_{-\nu(L)}^{L'}(1) \equiv 2^{\nu(L')}$ mod $2^{\nu(L')+1}$, so $q_{-\nu(L)}^{L'}(1) \equiv -2^{\nu(L)}$ mod $2^{\nu(L)+1}$. From 1), $q_{-\nu(L)-1}^{L_0^-} = 0$ and we are done. By induction $q_{-\nu(L)-1}^{L_0^+}(1) = q_{-\nu(L_0^+)}^{L_0^+}(1) \equiv 2^{\nu(L_0^+)}$ mod $2^{\nu(L_0^+)+1}$, so $q_{-\nu(L)-1}^{L_0^+}(1) \equiv 2^{\nu(L)+1}$ mod $2^{\nu(L)+2}$, or $q_{-\nu(L)-1}^{L_0^+}(1) \equiv 0$ mod $2^{\nu(L)+1}$ and 2) follows again.

Now we turn to 3). Note 3) holds for $r \geq \nu(L)$ but the result is vacuous. Nevertheless, we will not use $r < \nu(L)$ in the proof. From our formula we get $q_{-\nu(L)+2r}^L = -q_{-\nu(L)+2r}^{L'} -$

$q^{L_0}_{-\nu(L)+2r-1}$. By our induction hypotheses,

$$\begin{aligned}
q^{L'}_{-\nu(L)+2r} &= q^{L'}_{-\nu(L')+2r} = (\ell+\ell^{-1})^{\nu(L)-r}\ \tilde{q}^{L'}_{-\nu(L')+2r}\ . \\
q^{L_0^-}_{-\nu(L)+2r-1} &= q^{L_0^-}_{-(\nu(L_0^-)+1)+2r-1} = q^{L_0^-}_{-\nu(L_0^-)+2r-2} = q^{L_0^-}_{-\nu(L_0^-)+2(r-1)} \\
&= (\ell+\ell^{-1})^{\nu(L_0^-)-(r-1)}\ \tilde{q}^{L_0^-}_{-\nu(L)+2r-1} = (\ell+\ell^{-1})^{\nu(L)-r}\ \tilde{q}^{L_0^-}_{-\nu(L)+2r-1}\ . \\
q^{L_0^+}_{-\nu(L)+2r-1} &= q^{L_0^+}_{-(\nu(L_0^+)-1)+2r-1} = q^{L_0^+}_{-\nu(L_0^+)+2r} = (\ell+\ell^{-1})^{\nu(L_0^+)-r}\ \tilde{q}^{L_0^+}_{-\nu(L)+2r-1} \\
&=(\ell+\ell^{-1})^{\nu(L)+1-r}\ \tilde{q}^{L_0^+}_{-\nu(L)+2r-1} = (\ell+\ell^{-1})^{\nu(L)-r}\ \left((\ell+\ell^{-1})\tilde{q}^{L_0^+}_{-\nu(L)+2r-1}\right)\ .
\end{aligned}$$

Hence $(\ell+\ell^{-1})^{\nu(L)-r}$ divides each term on the right in our rewrite of (4) and hence divides $p^L_{-\nu(L)+2r}$.

Lastly, we turn to 4). Here it is easiest to work directly with (4), so

$$\begin{aligned}
P(L)(\ell,\ell+\ell^{-1}) =& -\ell^{-2\epsilon}P(L')(\ell,\ell+\ell^{-1}) - \ell^{-\epsilon}(\ell+\ell^{-1})P(L_0)(\ell,\ell+\ell^{-1}) \\
=&(-1)^{\nu(L)}\left(-\ell^{-2\epsilon}\cdot 1 - \ell^{-\epsilon}(\ell+\ell^{-1})\cdot(-1)\right) \\
=&(-1)^{\nu(L)}\ ,\ \text{since } \epsilon=\pm 1. \ \blacksquare
\end{aligned}$$

Remark: Both $c(L)$ and $u(L)$ depend on the projection. We can produce a link invariant by defining $C(L)$ to be the minimum over all regular projections for L of $c(L)$. We can also define $U(L)$ by minimizing $u(L)$. These are some of the easiest invariants of a link to define and are two of the most difficult to compute. The HOMFLY polynomial and other related polynomials were the first computable invariants to give some lower bounds for these invariants. Note that 1) implies $s \leq 2C(L) - \nu(L)$.

People have compiled tables of HOMFLY polynomials, but some care is needed in reading these tables. There is another polynomial also called the HOMFLY polynomial, which we shall denote by $\hat{P}$, and use variables a and z to distinguish it from P. This "other" HOMFLY polynomial satisfies

1) $\hat{P}(L)$ is an invariant of the oriented link L.
2) $\hat{P}(\text{unknot}) = 1$.
3) If L_+, L_- and L_0 are as usual,

$$a^{-1}\hat{P}(L_+) = a\hat{P}(L_-) + z\hat{P}(L_0) \tag{$\hat{*}$}$$

It is easy to translate from one polynomial to the other.

Theorem: *Let* $P(L) = \sum c_{r,s}\ a^r z^s$ *and* $\hat{P}(L) = \sum \hat{c}_{r,s}\ a^r z^s$. *Then*

$$c_{r,s} = (-1)^{\frac{-r+s}{2}}\ \hat{c}_{-r,s} \quad \text{and} \quad \hat{c}_{r,s} = (-1)^{\frac{r+s}{2}}\, c_{-r,s}$$

Proof: Define a Laurent polynomial $X_L(\ell,m) = \hat{P}(L)(i\ell^{-1}, im)$ where $i=\sqrt{-1}$. Then $\hat{*}$ becomes $\left(i\ell^{-1}\right)^{-1}X_{L_+} = (i\ell^{-1})X_{L_-} + (im)X_{L_0}$, or $(-i\ell)X_{L_+} = (i\ell^{-1})X_{L_-} + (im)X_{L_0}$, or

$0 = (i\ell)X_{L_+} + (i\ell^{-1})X_{L_-} + (im)X_{L_0}$. Divide by i to get $\ell X_{L_+} + \ell^{-1}X_{L_-} + mX_{L_0} = 0$. $X_{\text{unknot}} = \hat{P}(\text{unknot})(i\ell^{-1}, im) = 1$, so $X_L = P(L)$.

Hence $\sum c_{r,s}\ell^r m^s = \sum \hat{c}_{r,s}(i\ell^{-1})^r(im)^s$. But $\hat{c}_{r,s}(i\ell^{-1})^r(im)^s = \hat{c}_{r,s}i^{r+s}\ \ell^{-r}m^s$, so $\sum c_{r,s}\ell^r m^s = \sum \hat{c}_{-r,s}i^{-r+s}\ \ell^r m^s$. Since $r \equiv s$ mod 2, we can write $i^{-r+s} = (-1)^{\frac{-r+s}{2}}$. ∎

I found another set of tables (www.math.toronto.edu/stoimeno/poly.ps.gz) where the author used the relation

$$x^{-1}\dddot{P}(L+) + x\dddot{P}(L_-) = y\dddot{P}(L_0)\ .$$

Check that $c_{r,s} = (-1)^s\dddot{c}_{-r,s}$ for this choice.

For our next theorem using (4), we need a construction. Let L_1 and L_2 be oriented links. The link L_1 is equivalent to a link in the first octant so that the projection $(x, y, z) \to (x, y)$ is a regular projection. It lands in the first quadrant. Similarly, L_2 is equivalent to a link in the second octant so that the projection $(x, y, z) \to (x, y)$ is a regular projection landing in the second quadrant. Let $L_1 \amalg L_2$ denote the link obtained by taking the disjoint union. This extends our previous notation $L \amalg S^1$. Theorem 1 applies inductively to the case for which L_2 is an unlink on n components and yields $P(L_1 \amalg L_2) = \big(-(\ell + \ell^{-1})m^{-1}\big)^n P(L_1)$. Here is a generalization.

Theorem 10.

$$P(L_1 \amalg L_2) = \big(-(\ell + \ell^{-1})m^{-1}\big)P(L_1)P(L_2)\ .$$

Proof: For the writhes under our projection $(x, y, z) \to (x, y)$, we have $\omega(L_1 \amalg L_2) = \omega(L_1) + \omega(L_2)$, so it suffices to prove the formula for the Q polynomials. Fix the link L_1 and temporarily define $R(L) = Q(L_1 \amalg L)$. We need to prove $R(L) = A \cdot Q(L)$ where $A = \big(-(\ell + \ell^{-1})m^{-1}\big)Q(L_1)$.

If L is the regular projection of an unlink, then this formula follows from repeated applications of Theorem 1. If we apply (4) to $R(L)$ we see $R(L) = -R(L') - mR(L_0)$, which is the same as (4) applied to L and then multiplying by the constant A. ∎

We want to produce some examples of links with the same HOMFLY polynomial. To do this we describe a construction and a theorem. Define another operation on oriented links, $L_1 \# L_2$ as follows. Take $L_1 \amalg L_2$ with L_1 in the first octant, L_2 in the second and $(x, y, z) \to (x, y)$ a regular projection. Move L_2 straight down so it lies below the xy–plane (just add some large negative constant to the z–coordinate). Just to be safe, make it lie below the plane $z = -10$. Move L_1 straight up so it lies above the xy–plane and indeed above the plane $z = +10$.

Pick a point p_1 on L_1which is not a crossing. Consider the tangent line to this curve in $\mathbf{R}^3$ at the point p_1. Suppose the curve $\vec{r}$ is parameterized by arc length with $\vec{r}(0) = p_1$. If $\vec{v}$ is the derivative, $\vec{v} \neq 0$ (Why you 225 fans?). Since the projection is regular, the points where the projection of $\vec{v}$ to the xy–plane is 0 are isolated so choose p_1 so that the projection of $\vec{v}$ to the xy–plane is non–zero. Pick p_2 on L_2 similarly. By adding constants to the x and y coordinates of L_1 arrange for the projection of p_1 to the xy–plane to be the origin. Similarly slide L_2 so that the projection of p_2 is the origin.

Return to the discussion of L_1 and p_1. Let $\vec{w}$ denote the projection of the derivative at p_1 and recall $\vec{w} \neq 0$. The projection of p_1 to the xy–plane is the origin $\vec{0}$. Let $\hat{r}(s)$ denote the projection of $\vec{r}(s)$ to the same plane. An equation for the tangent line to $\hat{r}$ at the point $\vec{0}$ is $s \cdot \vec{w}$. Consider the function $F(s,t) = (1-t) \cdot \hat{r}(s) + t \cdot (s \cdot \vec{w})$ defined for all real numbers s and t and landing in the xy–plane. Note $F(s,0) = \hat{r}(s)$; $F(s,1) = s \cdot \vec{w}$ and $F(0,t) = \vec{0}$. Look at the projection of L_1 into the xy–plane. Near the origin we see a curve and we can restrict s to a sufficiently small interval, say $[-\epsilon_1, \epsilon_1]$ so that the projection of that segment goes through no crossings of the projection of L_1. Since $\vec{w} \neq 0$, either its x–component, $\vec{w}_x \neq 0$, or its y–component, $\vec{w}_y \neq 0$. If necessary, jiggle the embedding $L_1 \to \mathbf{R}^3$ a small amount to insure both $\vec{w}_x$ and $\vec{w}_y$ are non–zero. If necesaary, pick a smaller ϵ_1 so that for all $s \in [-\epsilon_1, \epsilon_1]$, both the x and y components of the derivative of $\hat{r}(s)$ never vanish. It follows that F is an embedding of $[-\epsilon_1, \epsilon_1] \times [0,1]$ into $\mathbf{R}^2$.

A *band* is just a short name for an embedded rectangle. Add a band to our segment which just drops straight down to $z = +10$. In the plane $z = 10$ we see the same curved segment that we saw in the xy–plane. Continue the band down to $z = 9$ using the formula $\big(F(s, z-9), z\big)$. The intersection of our band with the plane $z = 9$ is a straight line segment. Continue the band on down to the xy–plane as follows. One edge of our band is oriented since the segment from L_1 is and this determines an orientation on the whole band. Drop the point $(\vec{0}, 9)$ straight down. As you go, rotate the line segment about its center as necessary so that it intersects the plane $z = 8$ in a segment of the line of intersection of $z = 8$ with $x = 0$. In other words, in a segment of the line $(0, y, 8)$. Further arrange so that the preferred orientation on the segment points in the direction of increasing y. Now drop straight down to the xy–plane.

Add a band in a similar manner for $p_2 \in L_2$ except go up the z–axis and rotate so the segment of the line $(0, y, -8)$ points in the direction of decreasing y. Make one last adjustment to insure $\epsilon_1 = \epsilon_2$.

In the xy–plane the band coming down from L_1 and the band coming up from L_2 intersect in the same line segment so their union is a single band B. The band B intersects $L_1 \amalg L_2$ in two segments, one in L_1 and the other in L_2. Each of these segments is an entire side of B and they are opposite sides of the band. Call the one intersecting L_1 the "top" and the one intersecting L_2 the "bottom". The other two sides could be called "left" and "right" except this depends on which way you are viewing the band. However, the phrase *"left" and "right"* is well–defined. The link $L_1 \# L_2$ is the link obtained from $L_1 \amalg L_2 \cup B$ by deleting the interior of B and the open segments in $L_1 \amalg L_2$. Now $L_1 \# L_2$ minus the "left" and "right" sides of the band is identical to $L_1 \amalg L_2$ minus the segments from L_1 and L_2. The orientation on $L_1 \amalg L_2$ orients this subset and this orientation extends uniquely to $L_1 \# L_2$.

We turn to the uniqueness of this operation. First of all, with the links and the points fixed, we still can choose the width of the bands and can rotate them a little or a lot as we go to the xy–plane. Given two widths, we can choose a third which is smaller than both and if one width is smaller than the other, one of the bands is contained in the other and in this case the two sums are easily seen to be equivalent. Hence the width of the bands do not matter as long as they satisfy the requirements laid out above. If we do different rotations the net result is that the band is rotated be some number of full twists. A *full twist* is a rotation through ± 360 degrees. But given the link constructed using one band, the link constructed with that band rotated by a full twist is equivalent since we can rotate the link L_2 by 360 degrees which brings it back where it started.

A more informal description may be more convincing. Take models for L_1 and L_2. Take a rubber band and glue one end to L_1 at p_1 and the other end to L_2 at p_2. Put L_1 over L_2. You can put twists in the rubber band just by rotating the model for L_2, but you certainly aren't changing the link.

If we choose a different point on L_1 but still on the same component as our first choice, then we can shrink L_2 until it is very small and then use the band to draw it up until it is very close to the strand on L_1. Then we can slide the whole picture along the component until we get to our original point and then use the original band to drop L_2 back down and then expand it back to its original size. This shows that $L_1 \# L_2$ only depends on the component of L_1 in which we chose our point. A similar argument shows that it only depends on the component we chose in L_2. A variation on this argument shows that the order of the links does not matter either. If you have three links, a variation on this argument also shows $\big(L_1 \# L_2\big) \# L_3$ and $L_1 \# \big(L_2 \# L_3\big)$ are equivalent links as long as the same components are chosen in each link. Finally, if L_1 and L_1' are ambient isotopic, after we shrink L_2 into a small neighborhood of p_1, we can drag it along through the isotopy and then expand it again. This shows $L_1 \# L_2$ only depends on the equivalence classes of the two links and which two components we are adding together.

However, if L_1 has n components and L_2 has m, there are still potentially $n \cdot m$ different links represented by the symbol $L_1 \# L_2$. This is the connected sum of knots when both L_1 and L_2 are knots and since $1 \cdot 1 = 1$, it is an associative, commutative operation on equivalence classes of knots.

A knot K is called *prime* if whenever $K = K_1 \# K_2$, either K_1 or K_2 is trivial. The trivial knot is a unit for connected sum. It is a theorem that every knot K is a sum of prime knots and that the non–trivial knots which occur in a prime decomposition of K are unique up to permutation. In particular, the trivial knot is *the* unit for connected sum and no two non–trivial knots can sum together to give a trivial knot.

But the lack of uniqueness for multi–component links plus the next theorem will give us examples of different links with the same polynomial.

Theorem 11: $$P(L_1 \# L_2) = P(L_1) \cdot P(L_2).$$

Proof: In the construction on $L_1 \# L_2$, the projection to the xy plane is no longer regular. (We have those long vertical drops and ascents.) The projection into the yz–plane is regular in the region $-9 \le z \le 9$ and we can jiggle L_1 and L_2 separately until the projection of the whole link into the yz–plane is regular. Since we can always add full twists to the band, we can insure that in this projection there is at least one crossing between $z = -9$ and $z = 9$. If we switch this crossing we just have a band with a different number of twists and if we split at this crossing we just get a link equivalent to $L_1 \amalg L_2$. Hence $\ell P(L_1 \# L_2) + \ell^{-1} P(L_1 \# L_2) + m P(L_1 \amalg L_2) = 0$ and $P(L_1 \amalg L_2) = \big(-(\ell + \ell^{-1}) m^{-1}\big) P(L_1) P(L_2)$. Hence $(\ell + \ell^{-1}) P(L_1 \# L_2) = (\ell + \ell^{-1}) P(L_1) \cdot P(L_2)$ and the result follows since $\mathbf{Z}[\ell, \ell^{-1}, m, m^{-1}]$ is an integral domain. ∎

Remark: Theorem 11 says that P is multiplicative for the connected sum of knots.

We now turn to an invariant which will enable us to show that sometimes $L_1 \# L_2$ really does depend on which components are added. An n component link L is an embedding $\amalg_{i=1}^{n} e_i \colon \amalg\ S^1 \to \mathbf{R}^3$. Given a link L, we say L_1 is a *sublink* of L if there are integers $i_1, \ldots, i_r$ so that L_1 is the embedding $\amalg_{j=1}^{r} e_{i_j} \colon \amalg\ S^1 \to \mathbf{R}^3$. Loosely speaking, L_1 is a subset of L which either does not contain a component of L or else contains the entire component. If L is oriented so is each component of L and so L_1 is naturally oriented by restricting the orientation from the components of L.

Given given two oriented sublinks, L_1, $L_2 \subset L$ which are disjoint, define the *linking number*. $\mathbf{link}(L_1, L_2) \in \mathbf{Z}$ by summing the signs of the crossings for which one strand is in L_1 and the other is in L_2 and dividing by 2. There are several things to check in this definition, but first note that $\mathbf{link}(L_1, L_2) = \mathbf{link}(L_2, L_1)$ since we sum exactly the same numbers.

First check that $\mathbf{link}(L_1, L_2)$ is unchanged by Reidemeister moves. For type I moves this is obvious because the kinks introduced or deleted involve only a single strand. For type II moves, if both strings are in one sublink, the crossings make no contribution to $\mathbf{link}(L_1, L_2)$, whereas if they are in different sublinks one crossing contributes a $+1$ and the other a -1. For type III moves, the three strands could all belong to the same sublink and hence make no contribution to the linking number. Otherwise, two of the stands belong to one sublink and one belongs to the other. Without loss of generality, we may assume two of them belong to L_1 and one belongs to L_2. There are three ways to choose the strand belonging to L_2 and you can check that $\mathbf{link}(L_1, L_2)$ is unchanged in all three cases. This shows $\mathbf{link}(L_1, L_2)$ is a link invariant, but it is still not proved that it is an integer.

If we switch a crossing on L, either $\mathbf{link}(L_1, L_2)$ is unchanged or $\mathbf{link}(L_1, L_2)$ changes by ± 1. Crossing changes eventually unlink L and hence L_1 and L_2 and for these sublinks $\mathbf{link}(L_1, L_2) = 0$. Hence in general $\mathbf{link}(L_1, L_2) \in \mathbf{Z}$.

We record how $\mathbf{link}(L_1, L_2)$ changes under our various changes in a link. If we reverse all the orientations on L_1 and L_2, then $\mathbf{link}(L_1, L_2)$ is unchanged. If we reverse all the orientations on one sublink and keep them the same on the other, $\mathbf{link}(L_1, L_2)$ switches sign. It also switches sign if all the crossings in L are switched. Finally, a link with n components has n knot sublinks. If the subknots for L_1 are $K_1, \ldots, K_n$ and the subknots for L_2 are $K'_1, \ldots, k'_m$, then $\mathbf{link}(L_1, L_2) = \displaystyle\sum_{i=1}^{i=n} \sum_{j=1}^{j=m} \mathbf{link}(K_i, K'_j)$.

Here is an invariant of oriented links. It is most easily presented as a Laurent polynomial in a variable z. For each oriented link L with n components, let the subknots be $K_1, \ldots, K_n$ in any order. Then define

$$\mathcal{L}(L) = \sum_{i=1}^{n} z^{\mathbf{link}(K_i, L - K_i)} \ .$$

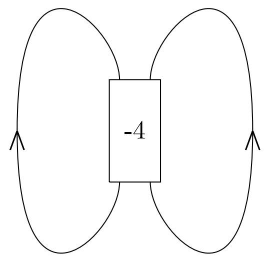
This invariant behaves a lot like the HOMFLY polynomial except it is much more primitive. It does not change if all orientations are reversed. Define bar in $\mathbf{Z}[z, z^{-1}]$ by sending z to

z^{-1} and extend to an automorphism. If L^* is the mirror image of L, $\mathcal{L}(L^*) = \overline{\mathcal{L}(L)}$. Hence if the oriented link is achiral then $\mathcal{L}(L) = \overline{\mathcal{L}(L)}$.

To describe our examples, let us take a minute to introduce an effective way of describing certain complicated knots and links. There is an example below of the general description coming next.

The diagram has some oriented arcs and some boxes. On the boundary of a box, we see four strands, two at one end of the box and two more at the other. The strands at one end are both going in and at the other end they are both going out. Inside each box is an integer.

To go from a diagram like this to a regular projection of a link, replace a box with integer n by two strands where the strands cross each other inside the region of the box $|n|$ times and are right handed crossings if $n > 0$ and left handed otherwise.



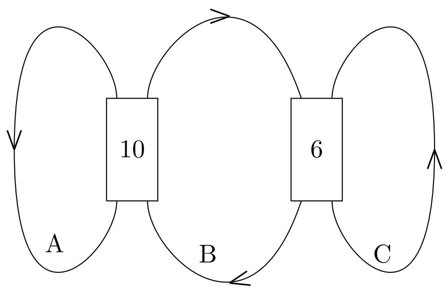
Since n is even, this regular projection is that of a link with 2 components. The linking number between these two components is $\frac{-4}{2} = -2$. Check that is we replace the -4 by ± 2 we get the right handed Hopf link for $+2$ and the left handed Hopf link for -2. We get the right handed trefoil for $+3$ and the left handed trefoil for -3. Note 0 gives the standard picture of the 2 component unlink and ± 1 give the two one crossing pictures of the unknot. Let $H\langle n\rangle$ be the knot or link with the same picture as above but with the integer n in the box. In particular, the picture is $H\langle -4\rangle$. Note that $H\langle n\rangle$ has some obvious symmetries. If we rotate $H\langle n\rangle$ 360 degrees around the vertical line through the box, we get $H\langle n\rangle$ back again but with the components switched. If we rotate by 360 degrees through the horizontal line through the box, we get $H\langle n\rangle$ back again except that all the orientations have been reversed. Finally, notice that the mirror image of $H\langle n\rangle$ is $H\langle -n\rangle$.

Exercise 1: Given that $P(H\langle 2\rangle) = -\ell^{-1}m + \ell^{-2}(\ell + \ell^{-1})m^{-1}$ and $P(H\langle 3\rangle) = \ell^{-2}m^2 - 2\ell^{-2} - \ell^{-4}$, compute $H\langle n\rangle$ for $n = 4$, 5 and 6.

Exercise 2: For $n > 0$ prove by induction that $P(H\langle n\rangle)$ is $(-1)^{n-1}\ell^{-n+1}m^{n-1}$ plus terms of order strictly smaller in m. Note that the theorem is false for $n = 0$ so you must start your induction with $n = 1$. Given the result for $n > 0$ prove (directly without induction) that for $n < 0$, $P(H\langle n\rangle)$ is $(-1)^{n-1}\ell^{-n-1}m^{-n-1}$ plus terms of order strictly smaller in m.

Remark: Notice that this proves that all the $H\langle n\rangle$ are distinct (except for $H\langle 1\rangle = H\langle -1\rangle$). I have no idea what it means, but note that the formula for $P(H\langle 0\rangle)$ is what you get by adding the answer you would get by assuming the formula for $n > 0$ worked for $n = 0$ to the answer you would get if the formula for $n < 0$ worked for $n = 0$.

To construct our examples, let L be the link below. It is a link with three components and the letters next to the strands serve merely to identify the components. Our examples are $H\langle -4\rangle \# L$. Notice that it does not matter which component of $H\langle -4\rangle$ we sum with because of the symmetry of $H\langle -4\rangle$.



Let L_A denote $H\langle -4\rangle \# L$ using component A from L; let L_B and L_C be defined similarly. Then $P(L_A) = P(L_B) = P(L_C)$ by Theorem 11. But $\mathcal{L}(L_A) = z^{-2} + 2z^3 + z^8$; $\mathcal{L}(L_B) = z^{-2} + z^3 + z^5 + z^6$; $\mathcal{L}(L_C) = z^{-2} + z + z^5 + z^8$, so all three links are distinct.

Let $H\langle p, q\rangle$ denote the 3 component link that results if the 10 in the figure is replaced by $2p$ and if 6 is replaced by $2q$. Hence the L of the last paragraph is $H\langle 5, 3\rangle$. Note $H\langle p, q\rangle = H\langle 2p\rangle \# H\langle 2q\rangle$ and we saw above that it doesn't matter which component is added to which for these particular links. You might want to wonder how the orientation on the left most component got switched from up to down. By rotating around the horizontal line through the middle, we see $H\langle p, q\rangle$ with the given orientation is equivalent to $H\langle p, q\rangle$ with all orientation switched. This gets the left most component oriented up, but now the right most one points down.

Exercise 3: There are nine ways to form the 5 component link $H\langle p, q\rangle \# H\langle r, s\rangle$. Can you find integers p, q, r and s so that all nine links are distinct? They do all have the same HOMFLY polynomial by Theorem 11. You might want to start by figuring out how to compute $\mathcal{L}(L_1 \# L_2) - \mathcal{L}(L_1) - \mathcal{L}(L_2)$ in terms of the two components being added.