Finite Order Equivalences of Embedded Graphs

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We discuss a notion of equivalence which is a bit different than the rigid equivalence we discussed earlier but is closely related to it. We will see that this relation, called *finite order equivalence*, is just what we need to exploit a big theorem of Flapan.

We need to explain two theorems from 3 manifold topology. The three manifold that is relevant to this discussion is S^3 which can be thought of in several ways. One way is that it is just $\mathbf{R}^3$ union a point ∞. Analogous examples which are easier to visualize are $\mathbf{R}^1 \cup \infty$ is the circle S^1 and $\mathbf{R}^2 \cup \infty$ is the two sphere S^2. All three of these examples can also be described as the unit sphere in Euclidean space of one higher dimension: $S^1 = \{x^2+y^2 = 1 \subset \mathbf{R}^2\}$; $S^2 = \{x^2+y^2+z^2 = 1 \subset \mathbf{R}^3\}$; $S^3 = \{x^2+y^2+z^2+t^2 = 1 \subset \mathbf{R}^4\}$.

Our first result is that any diffeomorphism $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ extends to a homeomorphism $\hat{f}\colon S^3 \to S^3$ defined as f on any point in $\mathbf{R}^3$ and $\hat{f}(\infty) = \infty$. If f is the identity so is $\hat{f}$. Since $\mathbf{R}^3 \subset S^3$ we can also think of h as an embedding $\hat{h}\colon G \to S^3$.

For notation, if f is a function from S^3 to itself, let $f^{(n)}$ denote the composition of f with itself n times. If $n = 0$ we define $f^{(0)}$ to be the identity and if f is a homeomorphism, we define $f^{(-n)}$ to be f^{-1} composed with itself n times. A homeomorphism $f\colon S^3 \to S^3$ has finite order provided that $f^{(n)} = f^{(0)}$ for some integer n.

We say two automorphism of an embedded graph $h\colon G \to S^3$, say θ_1 and θ_2, are *fo*–equivalent provided there exists a homeomorphism f of finite order such that $f \circ h \circ \theta_1 = h \circ \theta_2$.

The notion of *fo*–equivalence has some points in common with rigid equivalence and some points of divergence. Notice first that if θ_1 and θ_2 are rigidly equivalent via r, then r has finite order and so $\hat{r}$ shows that θ_1 and θ_2 are *fo*–equivalent. We could discuss a similar notion of *fo*–equivalence in which we replaced S^3 by $\mathbf{R}^3$ and again the $\mathbf{R}^3$ version would imply the S^3 version. We do not bother since Flapan's theorem only applies to the S^3 version and it is not true that *fo*–equivalent θ can necessarily be made equivalent by a finite order diffeomorphism of $\mathbf{R}^3$.

Perhaps the biggest difference between *fo* and rigid equivalence is that the set of θ *fo*–equivalent to the identity map of G do not form a subgroup. If θ is *fo*–equivalent to the identity, then so is θ^r for any r, but there is no obvious reason why $\theta_1 \circ \theta_2$ should be *fo*–equivalent to the identity even if both θ_1 and θ_2 are.

Before stating Flapan's theorem, we need a definition. Flapan calls a graph 3–connected if we can remove any 2 vertices from G, along with the edges incident to those vertices, and always have only one component.

Theorem 1:*(Flapan Theorem 6.1 p. 165) Let $h\colon G \to S^3$ be an embedding of a 3–connected graph. Let $f\colon S^3 \to S^3$ be a homeomorphism and θ an automorphism of G such that $f \circ h = h \circ \theta$. Equivalently, let $\theta \in \mathrm{Aut}_h(G)$. Then there exists a homeomorphism of finite order, $f_1\colon S^3 \to S^3$ and a new embedding $h_1\colon G \to S^3$ such that $f_1 \circ h_1 = h_1 \circ \theta$. If f preserves orientation, then f_1 can be chosen to preserve orientation and if f reverses orientation f_1 can be chosen to reverse orientation.*

Remark: We have no control over the f_1 or the h_1, but notice that the θ is the same.

Flapan's result is very powerful because of some theorems of P. A. Smith which say that the fixed point set of a finite order homeomorphism of S^3 is very restricted. It can be empty; it can be two points, or a circle (S^1), or an S^2 or an S^3. Note two points is the solution set to $x^2 = 1$ in $\mathbf{R}^1$ so we often write S^0 instead of two points. This way we can state Smith's theorem as: the fixed point set of a finite order homeomorphism of S^3 is either empty or an S^r for $r = 0$, 1, 2 or 3. Smith's theorem actually has a refinement of some relevance to us which says that the fixed set of f is empty, S^1 or S^3 if f is orientation preserving and S^0 or S^2 if f reverses orientation. Note that if the fixed set of an orientation reversing f is S^2, then $f^{(2)}$ is the identity since the fixed set of f is fixed by $f^{(2)}$ so its fixed set can only be S^3. There is a version for S^2 as well, which says that if f preserves orientation and has finite order, the fixed set is S^0; if f reverses orientation f has no fixed points or else S^1 is the fixed set.

Assume that there is at most one edge between any two vertices in G. If θ is an automorphism of G, let G^θ be the subgraph of G consisting of all vertices v such that $\theta(v) = v$ together with all edges in G both of whose endpoints are vertices of G^θ. This is not the whole story concerning what θ fixes. If θ has even order, there may be edges whose two endpoints are exchanged by θ.

Define $G^{[\theta]}$ to be G^θ plus one isolated vertex for each edge whose endpoints are interchanged. The number of vertices of $G^{[\theta]}$ can be worked out from the structure of the orbits of θ acting on the vertices of G. The number of vertices of G^θ is the number of orbits of this action with 1 element: the number of vertices of $G^{[\theta]}$ is at most the number of orbits with one or two elements. The difference between the number of vertices in $G^{[\theta]}$ and the number of vertices of G^θ is precisely the number of orbits with two vertices for which these two vertices are joined by an edge. Notice that if there are orbits with two elements, θ must have even order. If θ has order 2 and if there is at most one edge between any two vertices, then there are orbits with 2 vertices since every orbit has one or two vertices and if they all had one, θ would be the identity.

Define $G - Fix(\theta)$ to be the subgraph of G whose vertices are all the vertices of G which are not in G^θ: an edge of G is in $G - Fix(\theta)$ provided neither incident vertex is in G^θ nor are the two incident vertices exchanged by θ.

Because of the next two lemmas, the fixed set of f and $G^{[\theta]}$ are related. If f is a finite order homeomorphism of S^3, let $Fix(f)$ denote the subset of all points $x \in S^3$ with $f(x) = x$.

Lemma 2: *Let f be a homeomorphism from $[a, b]$ to itself. Suppose f has finite order. If f preserves orientation ($f(a) = a$), then f is the identity. If f reverses orientation ($f(a) = b$) then f is an involution and there is a unique point $x \in (a, b)$ which is fixed by f. Let $g\colon [a, x] \to [x, b]$ be the restriction of f. Then f restricted to $[x, b]$ is g^{-1}.*

A theorem from advanced topology called the Jordan–Brower separation theorem says that if you have an S^{n-1} embedded in any way in S^n, then $S^n - S^{n-1}$ divides into two pieces and any path from one piece to the other must cross the S^2. Applied to our situation it says

Lemma 3: *Let $f: S^3 \to S^3$ have finite order with $Fix(f)$ homeomorphic to S^2. Let x be any point in $S^3 - Fix(f)$. Then any path from x to $f(x)$ must cross the fixed set. If x and y can be joined by a path in $S^3 - Fix(f)$ then any path from x to $f(y)$ must cross the fixed set.*

Theorem 4: *Suppose there is at most one edge between any two vertices of G. Let $h: G \to S^3$ be an embedding and let $f: S^3 \to S^3$ be a finite order homeomorphism such that $f \circ h = h \circ \theta$. Then*

$$h\big(G^{[\theta]}\big) \subset Fix(f) \qquad \text{and} \qquad h\big(G - Fix(\theta)\big) \subset S^3 - Fix(f) \ .$$

Proof: By definition, all the vertices in G^θ map to $Fix(f)$. Consider any edge between such vertices. The homeomorphism f restricts to a homeomorphism which takes h of the edge to itself and fixes its endpoints. Of course the restricted homeomorphism has finite order, so by Lemma 2, h of the edge is in $Fix(f)$. Now consider an edge whose endpoints are exchanged. The homeomorphism f restricts to a homeomorphism of h of this edge which exchanges the endpoints. By Lemma 2 again, f restricted to h of this edge has a unique fixed point, $x \in Fix(f)$. Extend h to all of $g^{[\theta]}$ by sending the vertices in $G^{[\theta]}$ corresponding to these edges to the fixed points. This shows $h(G^{[\theta]}) \subset Fix(f)$.

Next note that all the vertices of $G - Fix(\theta)$ map via h into $S^3 - Fix(f)$ since θ moves all the vertices of $G - Fix(\theta)$. Now suppose h of some edge intersects $Fix(f)$. If one of the endpoints of this edge is mapped by h into $Fix(f)$, then this edge is not in $S^3 - Fix(f)$. Now suppose h of the edge intersects $Fix(f)$ in some interior point. Since h is an embedding, f must then map h of this edge into itself. If f fixes the endpoints of h of this edge then by Lemma 2, f fixes the entire edge so this edge is in G^θ and hence not in $G - Fix(\theta)$. If f exchanges the endpoints of h of this edge, then θ exchanges the vertices at the ends of the edge, so such an edge is not in $G - Fix(\theta)$ either. ∎

To use this theorem effectively, it will be good to have conditions on G (and not just on h) which guarantee that the image of h must be rather complicated. We state them for a general graph G but we will tend to apply them to G^θ or $G^{[\theta]}$.

Proposition 5: *If G has a vertex of valence at least 3, then there is no embedding of G into S^1.*

Proposition 6: *If G has a non–empty subgraph all of whose vertices have valence 2, then there is no embedding of G into $\mathbf{R}^1$.*

Also recall a theorem of Kuratoski.

Proposition 7: *If G contains a subgraph H which is a subdivision of K_5 or of $K_{3,3}$, then G has no embedding in S^2. Equivalently, if G is non–planar, G has no embedding in S^2.*

Here are detailed conclusions which can be drawn in the presence of finite order homeomorphisms whether they come from Flapan's Theorem or not.

Theorem 8: *Let G be a graph such that there is at most one edge between any two vertices and suppose G has a vertex of valence greater than 2. Let $h: G \to S^3$ be an embedding. Let $f: S^3 \to S^3$ be a finite order homeomorphism with $f \circ h = h \circ \theta$. Finally, suppose θ has order n.*

a) If f preserves orientation it has order n. If f reverses orientation and n is even, f has order n. Moreover, the following hold.

1) Suppose we are NOT in the case in which $n = 2$ and f reverses orientation. Then every vertex of G^θ has valence ≤ 2 and if there is one component of G^θ which is a circle, then this is all of $G^{[\theta]}$. If f reverses orientation then $G^{[\theta]}$ consists of 0, 1 or 2 vertices and no edges.

2) If $n = 2$, if f reverses orientation and if $G^{[\theta]}$ has at least 3 vertices or at least one edge, then G^θ has a planar embedding and no component of $G - Fix(\theta)$ is left invariant by θ.

b) If f reverses orientation and n is odd, f has order $2n$. Moreover G has a planar embedding. If $n > 1$ then G^θ is either empty or it consists of one or two vertices: it has no edges. If a vertex is not fixed by the θ action, then its orbit has n vertices in it.

Proof: Consider $F = f^{(n)}$ which has finite order since f does. Since θ^n is the identity, F fixes all of $h(G)$. If F preserves orientation, then F is the identity since a vertex of G has valence ≥ 3 so the fixed set of F can not be S^1 by Proposition 5. This proves F is the identity in the first two cases. If f reverses orientation and n is odd, F still fixes $h(G)$ but it reverses orientation so it is not the identity. Hence the fixed set of F is S^2 since it is not contained in S^0. Hence $F^{(2)}$ is the identity and G has a planar embedding.

This shows the order of f is no bigger than we expect, but why is it perhaps not smaller? Suppose $f^{(r)}$ were the identity with r properly dividing n. Then $f^{(r)} \circ h = h \circ \theta^r$ and θ^r is not the identity so $f^{(r)}$ does not even fix the image of h, much less all of S^3. Even in the last case $f^{(2r)}$ can not be the identity for r properly dividing n by the same argument. But $f^{(n)}$ can not be the identity either in our last case because, as we have already observed, $f^{(n)}$ reverses orientation.

To see part 1) of a), $G^\theta \subset Fix(f)$ by Theorem 4. If f preserves orientation, $Fix(f)$ is empty or S^1 and the result follows. If f reverses orientation and $n > 2$, apply this last remark to $f^{(2)}$ which has order $n/2 > 1$ and observe $G^\theta \subset G^{\theta^2}$. Since $Fix(f^{(2)})$ is S^1, $Fix(f)$ can not be S^2, so if f reverses orientation, $G^{[\theta]} \subset S^0$.

For part 2) of a) note $Fix(f)$ is S^0 or S^2 and since $G^\theta \subset Fix(f)$, $Fix(f) = S^2$. The restriction $G^\theta \subset Fix(f) = S^2$ shows the planar embedding.

Next, let $v \in G - Fix(\theta)$ be a vertex and suppose v and $f(v)$ lie in a single component of $G - Fix(\theta)$. Then there is a path in $h(G - Fix(\theta))$ from v to $f(v)$ and by Lemma 3 the path must cross the fixed set. But $G - Fix(\theta) \subset S^3 - Fix(f)$ and this is a contradiction. Hence v and $f(v)$ always lie in different components of $G - Fix(\theta)$, so no components are left invariant by θ.

To finish part b), let $X \subset S^3$ be the fixed set of $f^{(n)}$. Note X is homeomorphic to S^2. Check that $f(X) = X$. Since $f^{(n)}$ is the identity on X, f acting on X has finite order and we may think of h as an embedding $G \to X$. Two dimensional Smith theory applies here. Since n is odd and f has order n, f preserves orientation on X and Smith theory says the fixed set of this action is S^0 since f is not the identity. This says θ must fix one or two points since this set is a subset of S^0.

Now, let r be odd and suppose θ^r is not the identity. The above argument shows θ^r fixes one or two points and it clearly fixes all the points θ fixed. If θ fixes two points, θ^r fixes the same two points. If θ fixes one but θ^r fixes two, let x be the point fixed by θ^r but not by θ and let y be the point fixed by θ. Then x, y, $\theta(x)$, …, $\theta^{r-1}(x)$ are all points fixed by θ^r. Since there are only x and y, $\theta(x) = x$ or $\theta(x) = y$. Neither of these is possible. Hence $G^\theta = G^{\theta^r}$ which forces all orbits to either have one element or n. ∎

Remarks: The example of the tetrahedral embedding of K_4 in the plane shows that the last case can occur even with θ being the identity and hence having order 1.

Everything not forbidden by Theorem 8 can occur. You might like to think of some examples.

Corollary 9: *Let θ be an automorphism of G of order $n > 1$ and with $G^\theta = S^1$. Suppose G and $G - Fix(\theta) \neq \emptyset$ are connected. Then there is no embedding $h: G \to \mathbf{R}^3$ for which there exists a finite order homeomorphism $f: \mathbf{R}^3 \to \mathbf{R}^3$ such that $f \circ h = h \circ \theta$.*

Proof: Since $G - Fix(\theta) \neq$, $G \neq G^\theta$ and θ is not the identity. If all vertices of G have valence ≤ 2, then any edge–path (see the next section for a definition of edge–path) starting in G^θ stays in G^θ so G is not connected. Hence there are vertices of valence > 2.

Let $\hat{h}: G \to S^3$ and $\hat{f}: S^3 \to S^3$ be the results of adding the point at infinity to h and f. Note $\hat{f}$ has the same order as f and $\hat{f} \circ \hat{h} = \hat{h} \circ \theta$ and Theorem 8 applies to θ and $\hat{f}$. Note that $Fix(\hat{f})$ contains $G^{[\theta]}$ *and* one additional point, the point at infinity.

Suppose first we are in case a). If $n > 2$ or f preserves orientation then $Fix(f^{(2)})$ must be S^1 and contain $G^\theta = S^1$ plus ∞, which can not happen. If $n = 2$ and f reverses orientation, then $G - Fix(\theta)$ is disconnected. But we are assuming it is connected.

Hence we must be in case b) and $n > 1$. But then G^θ is a finite set of points which is a contradiciton. ∎

Three connected graphs.

Here are some results which will help us to show some graphs are 3 connected.

First of all, it is usually not too difficult to see that a graph is connected, but for a proof, "just" show that there is an edge–path between any two vertices. An *edge–path* between w_0 and w_1 is a sequence of edges, e_0, …, e_n such that e_0 is incident to w_0, e_n is incident to w_1 and for each i, $0 \leq i < n$, e_i and e_{i+1} share a vertex. Equivalently, one can write down a sequence of vertices beginning with w_0 and ending with w_1 such that any two adjacent vertices in the sequence have an edge between them. An edge–path is *embedded* provided all the vertices in this sequence are different. If two vertices can be joined by an edge–path, then they can be joined by an embedded edge–path. (If a vertex repeats simple delete all the vertices between two occurrences of it and note we still have an edge–path.) The *length* of an edge–path is the number of edges in it.

Lemma 10: *The number of edge–paths of length n in a graph is finite. If G has n vertices then if two vertices can be joined by an edge–path they can be joined by an edge–path of length $\leq n - 1$.*

Here is an algorithm to locate all the vertices which can be joined to v by an edge–path. Given any vertex v, let $\mathcal{S}(v)$ denote the set of vertices at the other end of the edges incident to v. Construct a non–decreasing sequence of set of vertices as follows. $S_0 = \{v\}$; $S_1 = S_0 \cup \mathcal{S}(v) = \bigcup_{w \in S_0} \mathcal{S}(w)$; $S_i = \bigcup_{w \in S_{i-1}} \mathcal{S}(w)$. As soon as $S_i = S_{i+1}$, S_i contains all the vertices in the same component as your initial v.

Lemma 11: *Let G be a connected graph and let $G_v \subset G$ be the subgraph with vertex v and all edges incident to v removed. Let $w(v)_1, \ldots, w(v)_r$ denote the vertices at the other end of the edges incident to v. Then G_v is connected if and only if $w(v)_i$ can be joined to $w(v)_{i+1}$ by an edge–path in G_v, $1 \leq i < r$.*

Proposition 12: *Let G_1 and G_2 be 3 connected subgraphs of G. Suppose $G_1 \cap G_2$ contains at least 3 vertices. Then $G_1 \cup G_2$ is 3 connected. If $G_1 \cap G_2$ has 2 vertices and one edge with one end in G_1 and the other in G_2, then $G_1 \cup G_2$ union this edge is 3 connected. If $G_1 \cap G_2$ has 1 vertex and two edges each with one end in G_1 and the other in G_2, then $G_1 \cup G_2$ union these two edges is 3 connected. If there are three edges each with one end in G_1 and the other in G_2, then $G_1 \cup G_2$ union these three edges is 3 connected.*

Proposition 13: *A graph G is r connected if and only if each pair of vertices can be joined by r embedded edge–paths which are distinct except for the initial and terminal vetrices.*

Proposition 14: *Let G_1 be a 3 connected subgraph of G. Let G_2 have the same vertices as G_1. Then G_2 is also 3 connected.*

Proposition 15: *Let G_1 and G_2 be 2 connected subgraphs of G. For each vertex of G_1 suppose there is at least one edge incident to that vertex whose other end is in G_2. Symmetrically, for each vertex of G_2 suppose there is at least one edge incident to that vertex whose other end is in G_1. Then $G_1 \cup G_2$ union these edges is 3 connected.*

Given two graphs G_1 and G_2 the *join* of G_1 and G_2, written $G_1 * G_2$ is the graph whose vertex set is the disjoint union of the vertex sets of the G_i. The edges consist of the edges of G_1, the edges of G_2 and one edge from each vertex of G_1 to each vertex of G_2. As an example, if G_1 and G_2 are both graphs with three vertices and no edges, $G_1 * G_2$ is $K_{3,3}$.

Proposition 16: *Suppose G_2 has at least 3 vertices. If G_1 has at least 3 vertices, $G_1 * G_2$ is 3 connected. If G_1 has 2 vertices and G_2 is conencted, $G_1 * G_2$ is 3 connected. If G_1 has 1 vertex and G_2 is 2 conencted, $G_1 * G_2$ is 3 connected.*

It follows from Proposition 16 that $K_{3,3}$ is 3 connected. Clearly K_3 is 2 connected and $K_n = K_{n-1} * \{v\}$, where $\{v\}$ is the graph with one vertex. By Proposition 16, K_n is 3 connected for $n \geq 4$.

We can now prove two more of Flapan's results. Recall that if an embedding $h\colon G \to \mathbf{R}^3$ is achiral, then there must be an automorphism θ and an orientation reversing diffeomorphism $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ such that $f \circ h = h \circ \theta$.

Theorem 17: *If G has no planar embeddings and if $h\colon G \to \mathbf{R}^3$ is an achiral embedding with automorphism θ, then θ has even order.*

Proof: As usual lately, let $h\colon G \to S^3$ and let $f\colon S^3 \to S^3$ reverse orientation with $f \circ h = h \circ \theta$. Assume to be contrary that θ has odd order r. Then $f^{(r)} \circ h = h \circ \theta^r = h$ and $f^{(r)}$ still reverses orientation since r is odd.

Since G is non–planar, by Kuratoski's theorem, Proposition 7, there is a graph H which is either $K_{3,3}$ or K_5 and some subdivision, H', of it so that H' is a subgraph of G. The embedding h restricts to an embedding of the subgraph H' which in turn yields an embedding $\hat{h}\colon H \to S^3$ and $f^{(r)} \circ \hat{h} = \hat{h}$.

Since H is 3 connected, Theorem 1 says we can construct a new embedding h_1 of H in S^3 and an orientation reversing, finite order homeomorphism f_1 of S^3 such that $f_1 \circ h_1 = h_1$. We can now apply Theorem 8 to conclude that the fixed set of the identity, which is H has a planar embedding, which is impossible. Hence θ must have even order. ∎

Flapan's Corollary 6.2, p.172, follows by applying Theorem 17 for θ the identity. Corollary 6.4 on page 176 follows since no such graph has an even order automorphism.

A few last remarks

We conclude the semester with

Theorem 18: *Any embedding of K_{4k+3}, $k > 0$, is chiral.*

Proof: If not, let θ be an automorphism exhibiting the achirality. By Theorem 17, if θ has order n, n is even. Suppose $n = 2^\ell \cdot r$ for r odd. Then the $f^{(r)}$ reverses orientation and θ^r exhibits the achirality: hence we may as well assume $n = 2^k$, $k > 0$. Consider θ acting on the vertices. There are a_1 orbits with 1 element, a_2 orbits with 2 elements, and a_{2^i} orbits with 2^i elements. The total number of vertices is then $\displaystyle\sum_{i=1}^{\infty} 2^i \cdot a_{2^i}$ which for us is $4k + 3 = a_1 + 2 \cdot a_2 + 4m$. Moreover, $G^{[\theta]}$ has $a_1 + a_2$ vertices since there is an edge between any two vertices.

Since K_{4k+3} is 3 connected, by Theorem 1 there is an achiral embedding $h\colon K_{4k+3} \to S^3$ and a finite order homeomorphism $f\colon S^3 \to S^3$ such that $f \circ h = h \circ \theta$.

Now apply Theorem 8. Since n is even we are in case a) and let us start by assuming that $k > 1$ so we are in subcase 1) with f reversing orientation. Since $G^{[\theta]}$ has 0, 1 or 2 vertices, it follows that $a_1 = a_2 = 1$. It further follows that $f^{(2)}$ preserves orientation and G^{θ^2} has three vertices and all the edges between them that G has. In this case G^{θ^2} is a triangle, a.k.a S^1. Hence $f^{(2^{k-1})}$ has order 2 and $G^{\theta^{2^{k-1}}}$ is S^1. But $G^{[\theta^{2^{k-1}}]}$ contains isolated vertices. This contradicts subcase 1) of a).

Could we be in subcase 2) of case a)? Since we can not be in subcase 1) and since K_{4k+3} has at least 6 vertices, $G^{[\theta]}$ has at least 3 vertices. Since G^{θ} is planar, θ can fix at most 4 vertices since if it fixed 5 it would fix a K_5 which is not planar. Hence $K_{4k+3} - Fix(\theta)$ has at least 3 vertices and since it has an even number of vertices, it has at least 4. But this forces $K_{4k+3} - Fix(\theta)$ to be connected: if x and y are vertices in $K_{4k+3} - Fix(\theta)$, there is an edge between them in K_{4k+3} and hence an edge between them in $K_{4k+3} - Fix(\theta)$ unless $f(x) = y$. But since there are at least 4 vertices, there is a vertex z in $K_{4k+3} - Fix(\theta)$ which is neither x nor $f(x)$ and so there is an edge in $K_{4k+3} - Fix(\theta)$ from x to z and from z to $f(x)$, so $K_{4k+3} - Fix(\theta)$ is connected. This contradicts the conclusion of subcase 2) since $K_{4k+3} - Fix(\theta)$ is supposed to have an even number of components. ∎

Flapan also shows that K_n does have achiral embeddings if n is not of the form $4k+3$, $k > 0$.