Rigid Equivalences of Embedded Graphs
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A *rigid motion* is a diffeomorphism of $\mathbf{R}^3$ which preserves distances and angles. A result from 3–dimensional geometry says that any rigid motion is a composition of translations, rotations around a fixed line and reflections in a fixed plane. If the motion has finite order, which means that some finite number of compositions of the motion with itself is the identity, then it is particularly simple. It is a composition of a rotation about a line and, possibly, a reflection through a plane perpendicular to the line. Reflections reverse orientation and translations and rotations preserve orientation.

We say that an embedded graph is *rigidly achiral* provided there is some rigid motion r which reverses orientation and some automorphism of the graph θ such that $r \circ h = h \circ \theta$. We can of course consider the case of *rigid equivalence*. Two embeddings $h: G \to \mathbf{R}^3$ and $h \circ \theta$ are *rigidly equivalent* provided there is a rigid motion r such that $r \circ h$ and $h \circ \theta$ are equal.

Rigid equivalence is a very strong condition. Since $r \circ h = h \circ \theta$, $r \circ h \circ \theta_1 = h \circ \theta \circ \theta_1$ for any automorphism θ_1. If there is also a rigid motion r_1 such that $r_1 \circ h = h \circ \theta_1$, then $(r \circ r_1) \circ h = h \circ (\theta \circ \theta_1)$ so $h \circ (\theta \circ \theta_1)$ is also rigidly equivalent to h. Furthermore $r^{-1} \circ h = h \circ \theta^{-1}$ so the set of all automorphisms of G with h rigidly equivalent to $h \circ \theta$ is a subgroup of $\mathrm{Aut}(G)$. For notation we use $\mathrm{Aut}_{rigid}(G, h)$. Note that it is possible for elements of $\mathrm{Aut}_{rigid}(G, h)$ to reverse orientation but every element of $\mathrm{Aut}_{rigid}(G, h)$ that preserves orientation is automatically in $\mathrm{Aut}_h(G)$. The set of elements of $\mathrm{Aut}_{rigid}(G, h)$ which preserve orientation is a subgroup of index at most 2, denoted $\mathrm{Aut}_{rigid^o}(G, h)$.

If the image of h is contained in a plane then reflections in that plane leave h fixed. If the image of h is a line, then even more rigid motions leave h fixed. But in the case where h lies in a lower dimensional subspace of $\mathbf{R}^3$ (a plane or a line or even a zero dimensional subspace if G is a single vertex), the correct question is how big a subgroup of the rigid motions of the subspace leave the image of h fixed. This has a uniform answer: the subgroup is the identity. If G is a single vertex this is obvious because the group of rigid motions is the identity. In the other cases, single out one vertex and in the 3 dimensional case, pick three additional points on the graph so that your four chosen points do not span a plane. Another way to say this is that if you look at the three vectors formed by going from your initial vertex to your other three chosen points, these three vectors are linearly independent. But since the graph is fixed, so are these three vectors and any rigid motion fixing your initial vertex is linear on vectors emanating from that vertex. Hence the space spanned by these three vectors is fixed. But this is all of $\mathbf{R}^3$ so your rigid motion is the identity. Similar, but easier, arguments establish the lower dimensional cases.

The group $\mathrm{Aut}_{rigid}(G, h)$ is a subgroup of $\mathrm{Aut}(G)$ and is hence finite. Given $\theta \in \mathrm{Aut}_{rigid}(g, h)$ we can also pick the r such that $r \circ h = h \circ \theta$ to have finite order as well. Indeed, we can pick it to have the same order as θ and even better, there is a homomorphism

$$\Phi: \mathrm{Aut}_{rigid}(G, h) \longrightarrow \mathcal{R}igid(3) ,$$

where $\mathcal{R}igid(3)$ denotes the group of rigid motions of $\mathbf{R}^3$, such that $\Phi(\theta) \circ h = h \circ \theta$. The subgroup $\mathrm{Aut}_{rigid^o}(G, h)$ lands in the orientation preserving rigid motions, a subgroup of

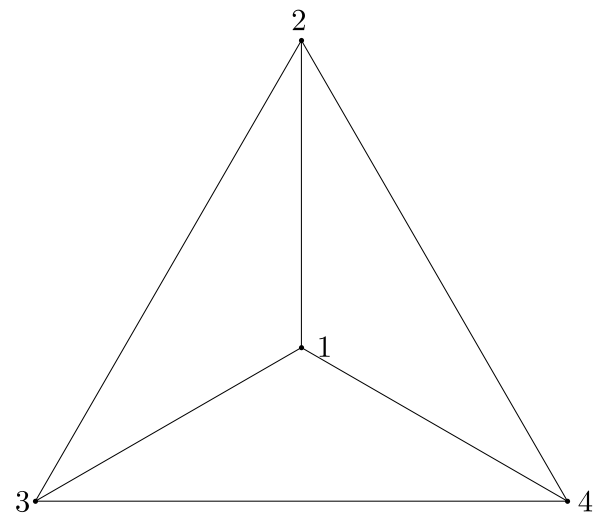
$\mathcal{R}igid(3)$ of index 2. If the image of h does not lie in a plane this is immediate since then the r such that $r \circ h = h \circ \theta$ is unique. In case the image of h is planar, we have chosen a vertex in the plane. Any rigid motion fixing this plane must also fix the line perpendicular to the plane through the vertex. There is a unique homomorphism $\mathrm{Aut}_{rigid}(G, h) \to \mathcal{R}igid(2)$ and we can embed $\mathcal{R}igid(2)$ into $\mathcal{R}igid(3)$ by insisting that the image fix this perpendicular line. A similar argument establishes the linear case and the case in which G is a single vertex. It follows that Φ is injective.

A theorem from advanced topology says that any finite subgroup of $\mathcal{R}igid(3)$ has at least one fixed point. This often makes it fairly easy to work out $\mathrm{Aut}_{rigid}(G, h)$ by first locating such a fixed point for $\Phi\big(\mathrm{Aut}_{rigid}(G, h)\big)$.

The group $\mathcal{R}igid(3)$ is the group relevant to solid Euclidean geometry and has been investigated for over three thousand years. It should perhaps not be so surprising that the finite subgroups are known. The finite subgroups of the oriented rigid motion group come in two infinite families and five exceptional families. One infinite family consists of the cyclic groups. The second infinite family consists of subgroups of the dihedral groups. These are all cyclic or dihedral except for one which is the Klein 4 group. This is the connected sum of two cyclic groups of order 2 and is sometimes referred to as the dihedral group of order 4. The five exceptional families are subgroups of the groups of symmetry of the five regular solids. Hence, for any embedded graph $h\colon G \to \mathbf{R}^3$, $\mathrm{Aut}_{rigid^o}(G, h)$ is one of the groups on this list. The finite subgroups of $\mathcal{R}igid(3)$ are all index 2 extensions of the list above. There is some overlap: for example the dihedral groups occur again as index 2 extensions of the cyclic groups, only now the involution that inverts the cyclic generator reverses orientation.

This rigidity makes $\mathrm{Aut}_{rigid}(G, h)$ very sensitive to h, unlike our previous ideas of "equivalence". For a generic embedding h, $\mathrm{Aut}_{rigid}(G, h)$ is the trivial group. For example, given any graph G with at most one edge between any two vertices and at least three vertices, you can always alter a given embedding of it by an arbitrarily small amount so that the distance between any two vertices is unique. For any such G and h, $\mathrm{Aut}_{rigid}(G, h)$ is trivial. On the other hand, human beings are good at recognizing symmetry and if you draw an embedding of G with a good deal of symmetry this often translates into a non–trivial $\mathrm{Aut}_{rigid}(G, h)$. For example, the obvious graph given by a picture of one of the five regular solids has the group of symmetries of that solid as its $\mathrm{Aut}_{rigid}(G, h)$.

In class we worked some examples with the symmetries of the cube. Here let us analyze the tetrahedron. Here is a picture of one looking down from the top vertex. Think of the triangle 234 as lying in the xy plane with the center of the triangle at the origin. Think of the vertex 1 as lying up the z axis. If you want actual points, let vertex 1 be $\left(0, 0, \sqrt{2}\,\right)$; vertex 2 be $(0, 1, 0)$; vertex 3 be $\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}, 0\right)$; vertex 4 be $\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}, 0\right)$.



There are some "obvious" symmetries. One is rotation by $\frac{2\pi}{3}$ about the z axis. There are actually two such rotations depending on whether we rotate clockwise or counterclockwise. The easiest way to describe either of these rotations is just to say what it does on the vertices. One rotation gives the permutation (234) and the other gives (243). Note $(234)^2 = (234)^{-1} = (243)$. There are also three reflections evident in this picture. Pick a vertex from $\{2, 3, 4\}$ and consider the plane perpendicular to the xy plane and containing the line from vertex 1 to your chosen vertex. Reflection in this plane is a symmetry. In terms of permutations, we get (34), (23) and (24), the transposition on the two elements we did not choose from $\{2, 3, 4\}$. It is not hard to convince yourself that this group is the dihedral group of order 6: the six elements being the identity and (234), (243), (34), (23) and (24). This is of course also the symmetric group on the three elements $\{2, 3, 4\}$.

Inside the symmetric group on 4 elements, we have 4 subgroups isomorphic to the symmetric group on 3 elements, namely the ones on $\{2, 3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 3, 4\}$. In particular, the symmetry group contains all transpositions and is therefore the whole group: Σ_4 is the group of symmetries of the regular tetrahedron.

There are rigidly achiral embeddings of both K_4 and K_6. Here is a picture of a rigidly achiral embedding of K_4.

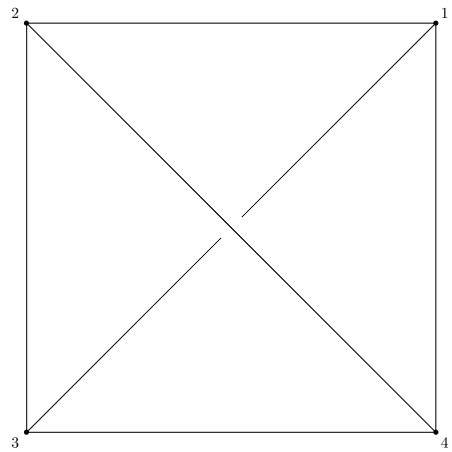


Figure K_4

Assume that you are looking down the z–axis and that the square is lying in the xy plane, $z = 0$, with the vertices at $(\pm 1, \pm 1, 0)$. Let the line from vertex 2 to vertex 4 be the two straight lines from the vertices to the point $(0, 0, 1)$ and let the line from vertex 1 to vertex 3 be the lines from the vertices to $(0, 0, -1)$.

This embedding is not planar so the rigid automorphism is determined by the automorphism of the graph. One such automorphism is seen to be (1234): if you send 1 to 2, 2 to 3, etc. the crossing at the center is reversed so you are looking at the mirror image. A formula for the rigid motion is $(x, y, z) \mapsto (y, -x, -z)$. This map is a linear map in the given coordinate system. It has determinant -1 so it reverses orientation. A linear map L is rigid if and only if $LL^{tr} = I$, where L^{tr} is the transpose of the matrix L and I is the identity matrix. Our map satisfies this condition so this embedding of K_4 is rigidly achiral. By definition $(1234) \in \mathrm{Aut}_{rigid}(K_4, h)$.

Another way to proceed is to note that if you apply (1234) then any fixed point of your rigid motion must be equidistant from the 4 vertices, which means it must lie on the line perpendicular to the center of the square. This line is the z axis and (x, y) goes to $(y, -x)$. But since the edges from 1 to 3 and from 2 to 4 consist of two line segments, the points $(0, 0, \pm 1)$ must be interchanged as well and this gives the formula.

It is also true that this embedding of K_4 is ambient isotopic to its mirror image. One possible ambient isotopy is to take the line between 1 and 3, which goes under the line from 2 to 4, and drag it to lie outside the square. Then drag it to lie above the line from 2 to 4. This is now the mirror image. It follows that $(1234) \in \mathrm{Aut}_h(K_4)$.

Another automorphism which shows the rigid achirality is (14)(23). This time the rigid motion is $(x, y, z) \mapsto (-x, y, -z)$. Just as before, this shows $(14)(23) \in \mathrm{Aut}_h(K_4)$ and $(14)(23) \in \mathrm{Aut}_{rigid}(K_4, h)$.

The permutations (1234) and (14)(23) generate the group of motions of the square, which is the dihedral group of order 8. The dihedral group of order 8 is the Sylow 2 subgroup of the symmetric group on 4 letters, which has order 24.

The permutation (12) is also in $\mathrm{Aut}_h(K_4)$ as the following pictures show.

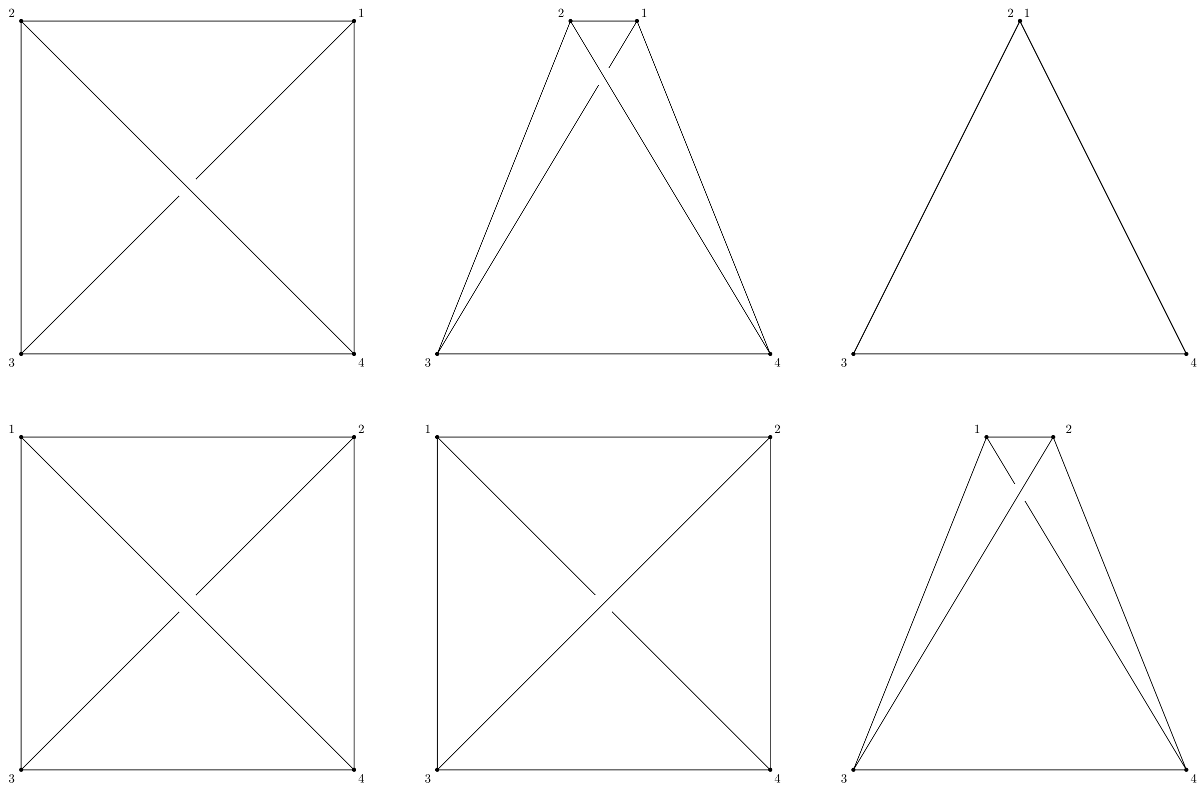


Figure K_4 A

One reads the pictures as follows. Assume the edges from 1 to 2 and from 3 to 4 are made of wooden sticks and let the other edges be rubber bands. Leave the edge from 3 to 4 in the plane and begin twisting the edge from 1 to 2 about its midpoint so that vertex 2 rises. Read the pictures beginning with the top row on the left. The middle picture shows vertex 2 rising and the right hand picture shows vertex 2 directly above vertex 1. Continue twisting with the picture on the far right of the bottom row and continue until the edge from 1 to 2 is flat again. This is the middle picture on the bottom row. Notice we have switched vertex 1 and 2 but we have the mirror image of $h \circ (12)$. But the mirror image is ambient isotopic to the original, so we have $h \circ (12)$, the final picture on the bottom row at the left.

Hence $(12) \in \mathrm{Aut}_h(K_4)$ and one can check that, since $\mathrm{Aut}_h(K_4)$ is a subgroup, $\mathrm{Aut}_h(K_4) = \mathrm{Aut}(K_4)$. Note $(1234)(12) = (234)$ so there are elements of order 3 in $\mathrm{Aut}_h(K_4)$ if you were worried.

The deformation from h to $h \circ (12)$ pictured above is certainly not a rigid motion. Indeed $(12) \notin \mathrm{Aut}_{rigid}(K_4, h)$ since the distance from vertex 1 to vertex 4 in h is 2 whereas in $h \circ (12)$ that distance is $2\sqrt{2}$. Notice in the real world that if you try to do the pictured deformation without allowing the rubber–band edges to expand and contract, the deformation will not work.

This last paragraph shows another feature of rigid equivalence. It is often comparatively easy to show that two embeddings are not rigidly equivalent whereas showing that two embeddings are not ambient isotopic is often rather difficult.

Another point that you may or may not have noticed is that the tetrahedron and K_4 are the same graph. Our embedding of the tetrahedron can be pushed into a plane (just think of vertex 1 as being the point $(0, 0, 0)$ and note that all the line segments lie in the xy plane). In this embedding, K_4 is achiral by a rigid motion which leaves the graph fixed. Of course the rigid motion given by Φ is the identity.

One further remark is that in the tetrahedral embedding, t, $(12) \in \mathrm{Aut}_{rigid}(K_4, t)$ where as we saw above that $(12) \notin \mathrm{Aut}_{rigid}(K_4, h)$ where h is the embedding from **Figure** K_4. Hence the rigid automorphism group is very sensitive to the embedding. Note that both t and h are ambiently isotopic. The diffeomorphism between these two embeddings can not be a rigid motion since if we apply a rigid motion to an embedding of a graph, the rigid automorphism subgroups are conjugate and for t and h this is not so.

Homework

Now consider the rigidly achiral embedding of K_6 given as follows. Place the vertices 1 through 4 in the xy plane as follows: vertex 1 at $(-1, 0, 0)$, vertex 2 at $(0, 1, 0)$, vertex 3 at $(1, 0, 0)$ and vertex 4 at $(0, -1, 0)$. Place vertex 5 at $(0, 0, 1)$ and vertex 6 at $(0, 0, -1)$. Run straight lines between all the pairs of vertices *except* 1 to 3 and 2 to 4. The edge from 1 to 3 is the straight line from vertex 1 to the point $(0, 0, 2)$ followed by the straight line from $(0, 0, 2)$ to vertex 3. The edge from 2 to 4 is the straight line from vertex 2 to the point $(0, 0, -2)$ followed by the straight line from $(0, 0, -2)$ to vertex 4.

Problem 1. Show this embedding is rigidly achiral. More specifically, write down an automorphism of this embedding of K_6 and write down a formula for the corresponding rigid motion.
Answer: There are several correct answers and the question is most easily dealt with as a part of question 3.

Problem 2. Using the handout sheet from class, or the copy on the web you can print with URL http://www.nd.edu/~taylor/Math468/K6.pdf, find all the maximal sublinks with non–zero linking number and explicitly verify that your automorphism takes a link with linking number n to a link with linking number $-n$, taking into account orientations.
Answer: $136 - 245$ has linking number $+1$ (the two crossings are both right handed). The other links have linking number 0 (five of them have no crossings and the other four have two which have opposite signs). We will finish this answer in question 3 as well.

Problem 3. Find all the elements in $\mathrm{Aut}_{rigid}(K_6, h)$.
Hint: Note that 1, 2, 3 and 4 lie in a plane and 5 and 6 are on a line perpendicular to this plane. Also the lines from 1 to 3 and 2 to 4 peak out along this line.
Answer: The z axis is the only line which contains an edge and intersects two other edges in their interiors, so it must be taken to itself by any rigid motion. The xy plane is perpendicular to the z axis and is the only such plane that contains four vertices, so it to must be left invariant by any rigid automorphism of the graph.

If the square 1, 2, 3, 4 is left fixed by a rigid motion then the line from 1 to 3 must also be left fixed (so must the line from 2 to 4). But this forces the z axis to be fixed.

Hence any rigid automorphism of this embedding is determined by what it does on the square 1, 2, 3, 4 so $\mathrm{Aut}_{rigid}(K_6, h)$ is a subgroup of the symmetries of the square. Indeed, there is a homomorphism which is injective $\mathrm{Aut}_{rigid}(K_6, h) \to D_8$.

Let $\theta = (13)(24)$. Check that if $r(x, y, z) = (-x, -y, z)$ $r \circ h = h \circ \theta$ and r is rigid. The symmetry of the square induced by θ is rotation by $180°$. To check that $r \circ h = h \circ \theta$ it is necessary and sufficient that we check the equation on the six vertices plus the two points $(0, 0, \pm 2)$. This follows because the rest of the embedding is given by straight lines between these points and rigid motions preserve straight lines.

If $\theta = (1234)(56)$, check that if $r(x, y, z) = (y, -x, -z)$, $r \circ h = h \circ \theta$ and r is rigid. The symmetry of the square induced by θ is rotation by $90°$.

Note $\theta^2 = (13)(24)$ so this gives another proof of the first result. Note this r reverses orientation since $\det \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1$. This is an answer to question 1. Note that $(1234)(56)$ takes $136 - 245$ to $245 - 316$ which has flipped the two components and reversed the orientation on one of them. Hence θ changes the sign of the linking number which is consistent with the result that r reverses orientation.

Furthermore $\theta^3 = (1432)(56)$ is also in $\mathrm{Aut}_{rigid}(K_6, h)$. As for the symmetries of the square, these are all the rotations.

Now turn to the reflections. Note $\theta = (13)$ reflects the square across the 2-4 diagonal. Check that if $r(x, y, z) = (-x, y, z)$, $r \circ h = h \circ \theta$ and r is rigid. Note (13) reverses orientation on $136 - 245$ and r reverses orientation so the linking numbers are consistent with the orientations (as they must be, this is just to reassure us), and we have another solution to question 1.

Let $\theta = (24)$. Check that if $r(x, y, z) = (x, -y, z)$ $r \circ h = h \circ \theta$ and r is rigid. The symmetry of the square induced by θ is reflection across the 1-3 diagonal. This is another solution to question 1.

Let $\theta = (12)(34)(56)$. Check that if $r(x, y, z) = (-x, y, -z)$ $r \circ h = h \circ \theta$ and r is rigid. The symmetry of the square induced by θ is reflection across the y axis.

Finally, let $\theta = (14)(23)(56)$. Check that if $r(x, y, z) = (-x, y, -z)$ $r \circ h = h \circ \theta$ and r is rigid. The symmetry of the square induced by θ is reflection across the x axis.

We have found eight elements in $\mathrm{Aut}_{rigid}(K_6, h)$ (the identity is in any group) and so the injective homomorphism $\mathrm{Aut}_{rigid}(K_6, h) \to D_8$ is an isomorphism and we have found all of $\mathrm{Aut}_{rigid}(K_6, h)$.

Problem 4. Show $(12) \notin \mathrm{Aut}_h(K_6)$. Show $(56) \notin \mathrm{Aut}_h(K_6)$.

Answer: Just compute the effect of (12) or (56) on $136 - 245$. Here is a more general argument that no transposition is in $\mathrm{Aut}_e(K_6)$ for any embedding $e \colon K_6 \to \mathbf{R}^3$. If $\theta \in \mathrm{Aut}_e(K_6)$, then $f \circ e = e \circ \theta$ for some homeomorphism of $\mathbf{R}^3$ to itself that preserves orientation, so all linking numbers must be preserved by θ if $\theta \in \mathrm{Aut}_e(K_6)$.

Since θ is a transposition, all the orbits of θ acting on the set of two component maximal sublinks have one or two elements. Every element in an orbit has the same linking number.

However, if a transposition fixes a maximal two component sublink, then it reverses the orientation on one of the components and leaves the orientation fixed on the other, so the linking number of $\theta(L)$ is minus the linking number of L. Hence any fixed link must have linking number 0 and so θ must not fix any L with odd linking number. But then there must be an even number of these. However, we know that there must be an odd number of them so we conclude $\theta \notin \mathrm{Aut}_e(K_6)$.

Just as a sanity check, note that we found two transpositions, (13) and (24), in $\mathrm{Aut}_{rigid}(K_6, h)$, but for both of these the r reversed orientation, so we did not show they were in $\mathrm{Aut}_h(K_6)$.