# Takehome Final for Math 468

## Solutions

1. We are given a continuous function $f\colon [a,b] \to [a,b]$ with $f(a) = a$, $f(b) = b$ and $f^{(n)}$ being the identity for some integer $n$. Out goal is to prove that $f$ is the identity. (You were actually told that $f$ was a homeomorphism, but this is redundant since if $f$ has order $n$, the inverse of $f$ is $f^{(n-1)}$.)

   Now $f$ is the identity if and only if $f(x) = x$ for all $x \in [a,b]$ so it behoves us to study the subset $C = \big\{x \in [a,b] \,\big|\, f(x) = x\big\}$ and its compliment $A = \big\{x \in [a,b] \,\big|\, f(x) \neq x\big\}$. It will also be good to divide $A$ into two disjoint sets, $A_+ = \big\{x \in [a,b] \,\big|\, f(x) > x\big\}$ and $A_- = \big\{x \in [a,b] \,\big|\, f(x) < x\big\}$.

   Since $f$ is continuous, the set $C$ is closed. Proof: if $x_n$ is a sequence in $C$ converging to $x$ then $x \in [a,b]$ (because $[a,b]$ is closed) and $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x$. But this means $x \in C$, so $C$ is closed.

   The set $C \cup A_+ = \big\{x \in [a,b] \,\big|\, f(x) \geq x\big\}$ is also closed. Proof: if $x_n$ is a sequence in $C \cup A_+$ converging to $x$ then $x \in [a,b]$ ($[a,b]$ is closed) and $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) \geq \lim_{n\to\infty} x_n = x$. But this means $x \in C \cup A_+$, so $C \cup A_+$ is closed. Hence $A_- = [a,b] - (C \cup A_+)$ is open. A similar argument shows $A_+$ is open.

   One of the hints then says that if $A_+$ is not empty, $A_+ = (x_0, x_1) \cup \tilde{A}$, where $\tilde{A}$ is open and the union is disjoint. Since the union is disjoint, $x_0$ and $x_1$ are not in $A_+$. Proof: If $x_0 \in A_+$ then $x_0 \in (y_0, y_1) \subset \tilde{A}$ for some open interval. (The hint again.) But then $(y_0, y_1) \cap (x_0, x_1)$ is not empty. A similar contradiction shows $x_1 \notin A_+$.

   But since $C \cup A_+$ is closed, $x_0$ and $x_1$ must be in $C \cup A_+$ and therefore they must be in $C$. In other words $f(x_0) = x_0$ and $f(x_1) = x_1$. Since $f$ is continuous, $f$ assumes a maximum value, say $M$ at a point $y \in [x_0, x_1]$. Since $f(x_1) = x_1$, $M \geq x_1$. If $M > x_1$, then by the Intermediate Value Theorem applied to $f$ on the interval $[x_0, y]$, there is some $y_1 \in [x_0, y]$ such that $f(y_1) = x_1$ so $f$ is not one–to–one. Hence $M = x_1$ and for any $y \in [x_0, x_1)$, $f(y) < x_1$. A similar argument shows $x_0$ is the minimum value assumed by $f$ on $[x_0, x_1]$ so $f$ restricts to a finite order homeomorphism $f\colon [x_0, x_1] \to [x_0, x_1]$ and moreover $f$ takes $(x_0, x_1)$ into $(x_0, x_1)$.

   Pick any point $y \in (x_0, x_1)$. Then $f(y) \in (x_0, x_1)$ and $y < f(y)$. Applying $f$ to $f(y)$ shows $f(y) < f\big(f(y)\big) = f^{(2)}(y)$ and hence $y < f^{(2)}(y)$ and $f^{(2)}(y) \in (x_0, x_1)$. By induction, if $y < f^{(n-1)}(y)$ and $f^{(n-1)}(y) \in (x_0, x_1)$, then applying $f$ to $f^{(n-1)}(y)$ shows $f^{(n-1)}(y) < f\big(f^{(n-1)}(y)\big) = f^{(n)}(y)$ and $f\big(f^{(n-1)}(y)\big) \in (x_0, x_1)$. Hence $y < f^{(n)}(y)$ and $f^{(n)}(y) \in (x_0, x_1)$. But for some $n$, $f^{(n)}(y) = y$ and this is a contradiction. Hence $A_+ = \emptyset$.

   A similar argument shows $A_- = \emptyset$. Therefore $C = [a,b] - (A_+ \cup A_-) = [a,b]$ which is what we set out to prove.

2. Here the problem is to show that you can not embed three intervals into the real line so that their initial points all go to the same value but this is the only common point. More elaborately, we have three intervals $I_i = [0,1]$ $i = 1$, 2 and 3 and three one to one maps $f_i\colon [0,1] \to (-\infty, \infty)$ such that $f_1(0) = f_2(0) = f_3(0) = a$ and if $f_i(x) = f_j(y)$ for $i \neq j$ then $x = y = 0$. Moreover, each $f_i$ is an embedding although all we will need of this hypothesis is that $f_i(1) \neq f_i(0)$ for all three $i$. Let $x_i = f_i(1)$ and note that none of the $x_i$ are equal to $a$.

   By hypothesis, the three $x_i$ are distinct. There are two cases.

   Case 1. At least two of the $x_i$ are bigger than $a$.

   Case 2. Otherwise at least two of the $x_i$ are less than $a$.

   In case 1, we can renumber the intervals and $f$'s if necessary so that $x_1$ and $x_2$ are greater than $a$. Since $x_1 \neq x_2$ we can also renumber so that $x_1 < x_2$. Now $f_2(0) = a$ and $f_2(1) = x_2$ so by the Intermediate Value Theorem there is some $t \in [0,1]$ such that $f_2(t) = x_1 = f_1(1)$ and this is a contradiction.

   A similar argument shows case 2 can not happen either.

3. The automorphism group of the graph $K_6$ is the symmetric group on 6 elements, namely the vertices. The maximal sublinks determine and are determined by a pair of subsets of $\{1,2,3,4,5,6\}$, each subset having 3 elements (so there are $\frac{1}{2}\cdot\binom{6}{2} = 10$ of them). Any element in the symmetric group acts on the set of links: if $\theta$ is the permutation and $L = \big\{\{x_1,x_2,x_3\},\{x_4,x_5,x_6\}\big\}$ is the maximal sublink then

$$\theta(L) = \big\{\{\theta(x_1),\theta(x_2),\theta(x_3)\},\{\theta(x_4),\theta(x_5),\theta(x_6)\}\big\}\ .$$

   The first step in working this problem is to solve two cases of a general combinatorial problem: given the symmetric group on a set, how do the elements of it act on the set of pairs of three–element subsets. For us it will be sufficient to work out the orbit structure of this action for any element of order 5 and any element of order 3. Notice that this part of the problem has nothing to do with any embedding of the graph $K_6$.

   Here are some useful general remarks. Suppose that $\theta$ fixes $L$, so $L = \theta(L)$. For notation, let $\{x_1,x_2,x_3\} = S_1$ and $\{x_4,x_5,x_6\} = S_2$ so $L = \big\{S_1,S_2\big\}$. Look at $\theta(x_1)$: either $\theta(x_1) \in S_1$ or $\theta(x_1) \in S_2$ and if $\theta(x_1) \in S_i$, $\theta(x_2)$ and $\theta(x_3) \in S_i$ as well. Furthermore, $\theta(x_4)$, $\theta(x_5)$ and $\theta(x_6)$ are all in the other $S_j$. So if $\theta(x_1) \in S_2$, then $\theta^2(x_1) \in S_1$. If additionally $\theta$ has order $2r+1$, then $\theta(x_1) \notin S_2$ since $\theta^{2r}(x_1) \in S_1$ so $\theta^{2r+1}(x_1)$ would be in $S_2$ but since $\theta^{2r+1}$ is the identity, $\theta^{2r+1}(x_1) = x_1$ would also be in $S_1$. Hence odd order permutations that fix an $L$ also fix the $S_1$ and the $S_2$.

   Just to be pedantic, note that we say $\theta$ fixes a subset provided $\theta$ takes each element in the subset to another element in the subset: $\theta$ need not fix every element in the subset.

Hence if $\theta$ fixes $S_1$ then $\theta(x_1) = x_1$, $x_2$ or $x_3$. If $\theta(x_1) = x_1$, then $\theta(x_2) = x_2$ and $\theta$ fixes each element of $S_1$ or else $\theta(x_2) = x_3$. But then $\theta^2(x_2) = x_2$ and if $\theta$ has odd order this is again a contradiction. So if $\theta$ has odd order and fixes $S_1$, either $\theta$ fixes every element of $S_1$ or $\theta(x_1) \neq x_1$ and $\theta^3$ fixes every element of $S_1$. Of course there is a similar analysis for $\theta$ acting on $S_2$.

We do a $\theta$ of order 5 first. We claim any such $\theta$ fixes exactly one element of $\{1, 2, 3, 4, 5, 6\}$. This is a baby version of the main argument. Look at the orbits of this action. Since 5 is prime, orbits have either 1 or 5 elements. In our case we have 6 elements so we either have 1 one–element orbit and 1 five–element orbit or we have 6 one–element orbits. But 6 one–element orbits means that $\theta$ is the identity whereas we assumed it had order 5.

Now suppose $\theta(L) = L$. Since 5 is odd, $\theta(S_i) = S_i$. Since $\theta^3$ fixes each element of $S_i$, so does $\theta^6$. But $\theta^6 = \theta$ so $\theta$ fixes all six elements of $\{1, 2, 3, 4, 5, 6\}$, contradiciton. Hence $\theta$ fixes no sublink so all the orbits must have 5 elements since 5 is a prime. Therefore there are 2 five–element orbits.

Now we analyze the case(s) in which $\theta$ has order 3. Since 3 is also prime, orbits either have one element or three. Hence there are two types of permutations in the symmetric group on six elements: those which have 3 one–element orbits and 1 three–element orbit for the action of $\theta$ on $\{1, 2, 3, 4, 5, 6\}$ and those which have 2 three–element orbits. (Again, just like for 5, 6 one–element orbits means you have the identity.)

Three is odd so in this case too, $\theta(S_i) = S_i$ if $\theta$ fixes $L = \{S_1, S_2\}$. Again either $\theta$ fixes each element in $S_i$ or fixes none, but it can not fix all the elements in both since then $\theta$ would be the identity. Unlike the order 5 case, this can definitely happen for order 3. But from the last paragraph, a $\theta$ of order 3 fixes either 3 elements of no elements and in either case the is at lest one orbit $A \subset \{1, 2, 3, 4, 5, 6\}$ with three elements. But the $A$ must equal $S_1$ or $S_2$ and since $S_1$ determines $S_2$ and vice versa, $\theta$ fixes exactly one $L$. Since there is 1 one–element orbit and since 3 is prime, there must be 3 three–element orbits.

So far we have just been describing how various permutations act on the set of maximal sublinks. Now suppose there is an embedding $h\colon K_6 \to \mathbf{R}^3$. If $\theta \in \mathrm{Aut}_h(K_6)$ and if $L$ is a maximal sublink, then $L$ and $\theta(L)$ have the same linking number up to sign. If we are careful, we can even figure out if the two linking numbers are the same or if they are the negatives of each other. In particular, the orbits of $\theta$ either have all linking numbers even or all of them are odd.

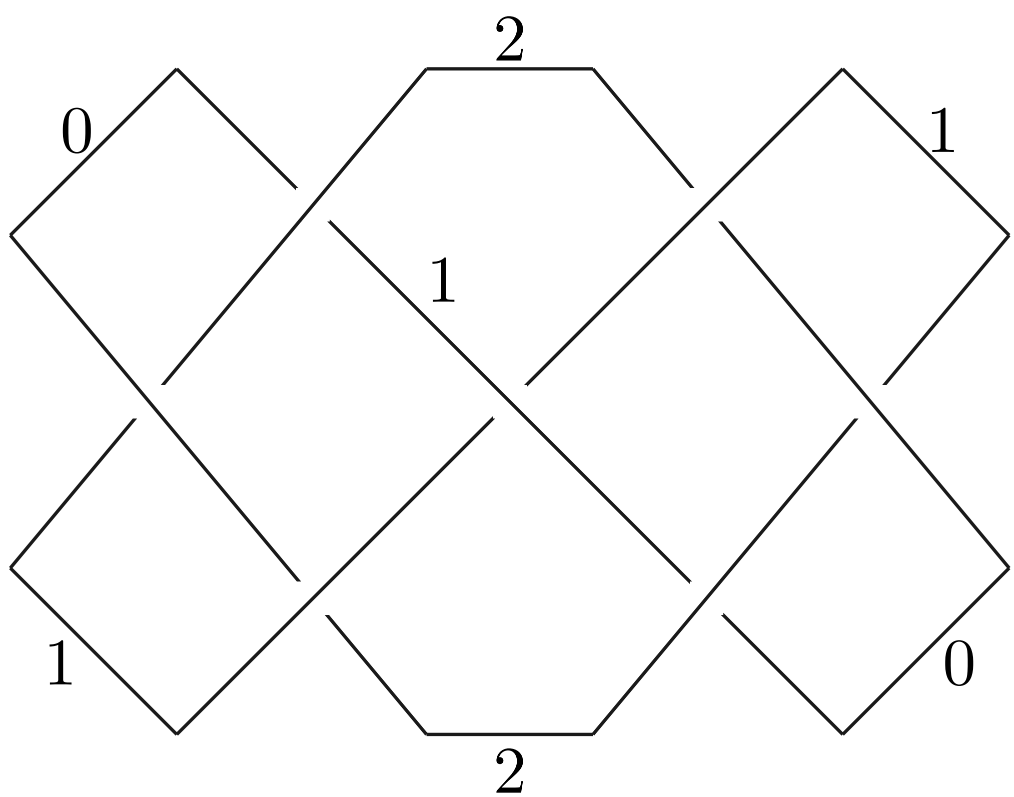
Finally suppose $\theta_1 \in \mathrm{Aut}_h(K_6)$ has order 5 and $\theta_2 \in \mathrm{Aut}_h(K_6)$ has order 3. The $\theta_1$ action shows there are 5 sublinks with odd linking number and 5 with even linking number. The $\theta_2$ action shows that there is either 1 sublink with odd linking number, or there are 3 of them, or there are 7 of them, but in no case are there 5 of them.

4. The easiest way to show a knot has a 3 coloring is to simply exhibit one. In this case I found one by assign 1 to the first segment, which I took to be the lower left segment with

slope $-1$ and went up. Went I got to the first crossing, I just assigned the next segment a number of 2 and then $3 = 0$ when I got to the next crossing and then 1 after the next crossing.

Now I am at the upper left crossing and here two of the three segments at the crossing are already assigned. But we can certainly assign a number to the third segment so that the required equation holds. This idea works to assign a unique number to each remaining segment until we arrive at the crossing in the center. Here you have no more free edges so either the relation holds or it doesn't. Fortunately for me, the relation does hold.

You can also prove the *existence* of a 3–coloring by computing a $6 \times 6$ determinant.

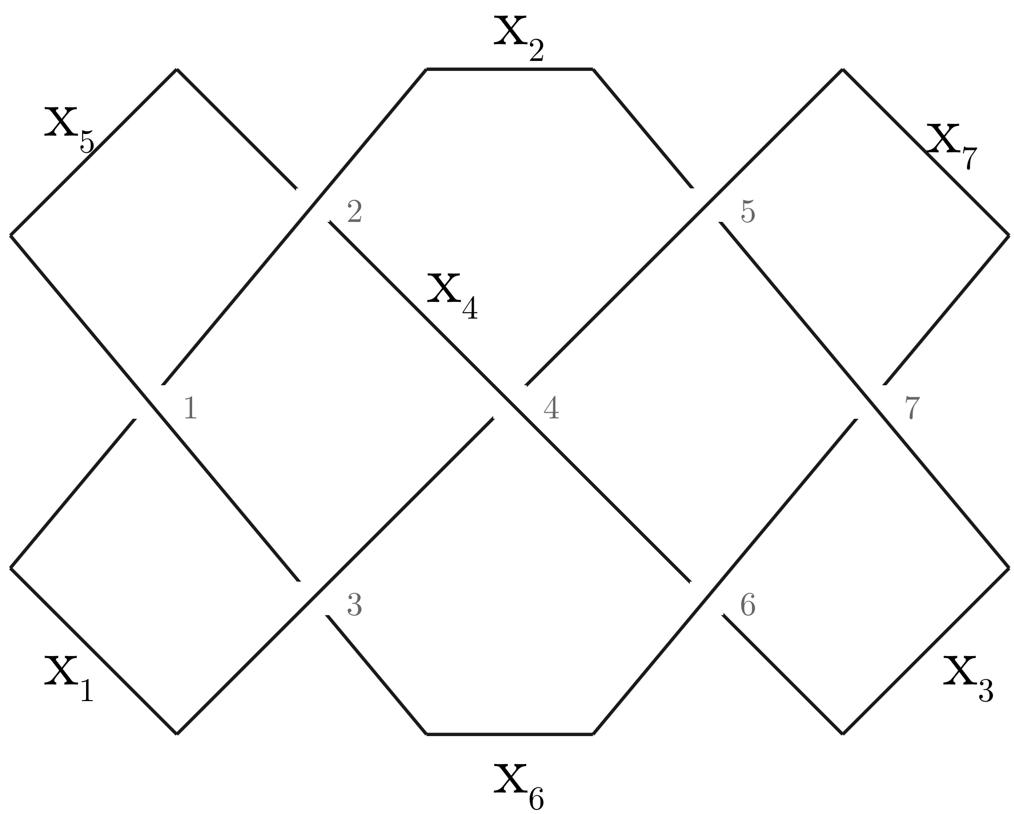



Writing generators and relations for the universal group is fairly straight forward. First you need to orient the knot and decide which crossings are right handed and which are left handed. We also need to assign variable names to the segments. We assign names as indicated in the next picture and orient the knot so that if we start at $x_1$ we go in the direction of $x_2$. No matter how we orient the knot, all the crossings are right handed.

Hence the group is the group generated by elements $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ subject to the seven relations where the number at the crossing is the number of the relation:

| | | | |
|---|---|---|---|
| 1. $x_2 = x_5 x_1 x_5^{-1}$ | 2. $x_5 = x_2 x_4 x_2^{-1}$ | 3. $x_6 = x_3 x_5 x_3^{-1} 2$ | 4. $x_1 = x_4 x_7 x_4^{-1}$ |
| 5. $x_3 = x_7 x_2 x_7^{-1}$ | 6. $x_4 = x_6 x_3 x_6^{-1}$ | 7. $x_7 = x_3 x_6 x_3^{-1}$ | |

You weren't asked, but note the relation between the universal knot group and the coloring. Arbitrarily assign colors to transpositions in the symmetric group on three letters, $\Sigma_3$: say $1 \mapsto (23)$, $2 \mapsto (12)$ and $0 \mapsto (13)$. If segment $x_i$ is colored with $k$ and if $k \mapsto (ab)$, define a function $\psi\colon G \to \Sigma_3$ by $\psi(x_i) = (ab)$. Check that seven relations hold. Here is what you have to check for 1: $\psi(x_2) = (12)$, $\psi(x_1) = (13)$, $\psi(x_5) = (23)$ AND $(12) = (13)(23)(13)$. The general theory then says that $\psi$ is a group homomorphism and it is easy to check that in this case it is onto.




Finally we need to compute the HOMFLY polynomial. Most of the work involves finding crossings we can change to get to knots and links that are simple enough that we recognize them. Pages 7 through 10 show one such path. The page 7 is just the knot, labeled $K$. In that picture, one crossing is labelled with a + to signify two things. First of all it is a right handed crossing since all the crossings are right handed and second, this is the crossing we will switch. In our HOMFLY skein formula,

$$\ell P(L_+) + \ell^{-1} P(L_-) + m P(L_0) = 0 \ ,$$

$K = L+$. On page 8 we draw $K1$, which is $K$ with the crossing switched, and $K0$ which is $K$ with the crossing split. Hence $L_- = K1$ and $L_0 = K0$.

Neither $K1$ nor $K0$ is immediately recognized by me, so each of them has a plus to indicate the crossing is right handed and that I wish to apply the skein formula to that particular crossing. Since $K0$ is a link, actual orientations are important, so orient $K$ the same way we have been and then note that there are evident orientations on $K1$ and $K0$.

Page 9 ignores $K0$ and continues the saga of $K1$. When we apply the skein formula to the indicated crossing, $L_+$ is $K1$, $L_-$ is labelled $K11$ and $L_0$ is labelled $K10$. The knot $K11$ is also labelled as "unknot" since it can be seen to be unknotted. I still don't recognize $K10$ so it has another marked crossing and page 10 gives the associated links: $K10$ will be $L_+$ again since the crossing is $+$; $L_1$ will be $K101$ and $L_0$ will be $K100$. Both of these I recognize and are named.

On page 11 we return to $K0$ from page 8. Again $L_+$ is $K0$ and $L_1$ is labelled $K01$ and $L_0$ is labelled $K00$. Both of these I recognize and have named.

We are now in business. Since all the crossings we switched are right handed, we can solve the skein relation once and for all as

$$P(L_+) = -\ell^{-2}P(L_-) - \ell^{-1}mP(L_0)\ .$$

Also we note that only the +Hopf link and the unknot occur on our list of recognized links, so for now let $H$ be the HOMFLY polynomial for the +Hopf link. Since the HOMFLY polynomial for the unknot is 1 it hardly seems worthwhile renaming it.

Let us first compute $P(K0)$: we have $P(K0) = -\ell^{-2}P(K01) - \ell^{-1}mP(K00) = -\ell^{-2}H - \ell^{-1}m \cdot 1 = -\ell^{-2}H - \ell^{-1}m$.

Note $K10$ and $K0$ are the same link, so $P(K10) = P(K0)$.

$P(K1) = -\ell^{-2}P(K11) - \ell^{-1}mP(K10) = -\ell^{-2} - \ell^{-1}m\big(-\ell^{-2}H - \ell^{-1}m\big) = -\ell^{-2} + \ell^{-2}m^2 + \ell^{-3}mH$.

Finally $P(K) = -\ell^{-2}P(K1) - \ell^{-1}mP(K0) = -\ell^{-2}\big(-\ell^{-2} + \ell^{-2}m^2 + \ell^{-3}mH\big) - \ell^{-1}m\big(-\ell^{-2}H - \ell^{-1}m\big) = \ell^{-4} - \ell^{-4}m^2 - \ell^{-5}mH + \ell^{-3}mH + \ell^{-2}m^2$

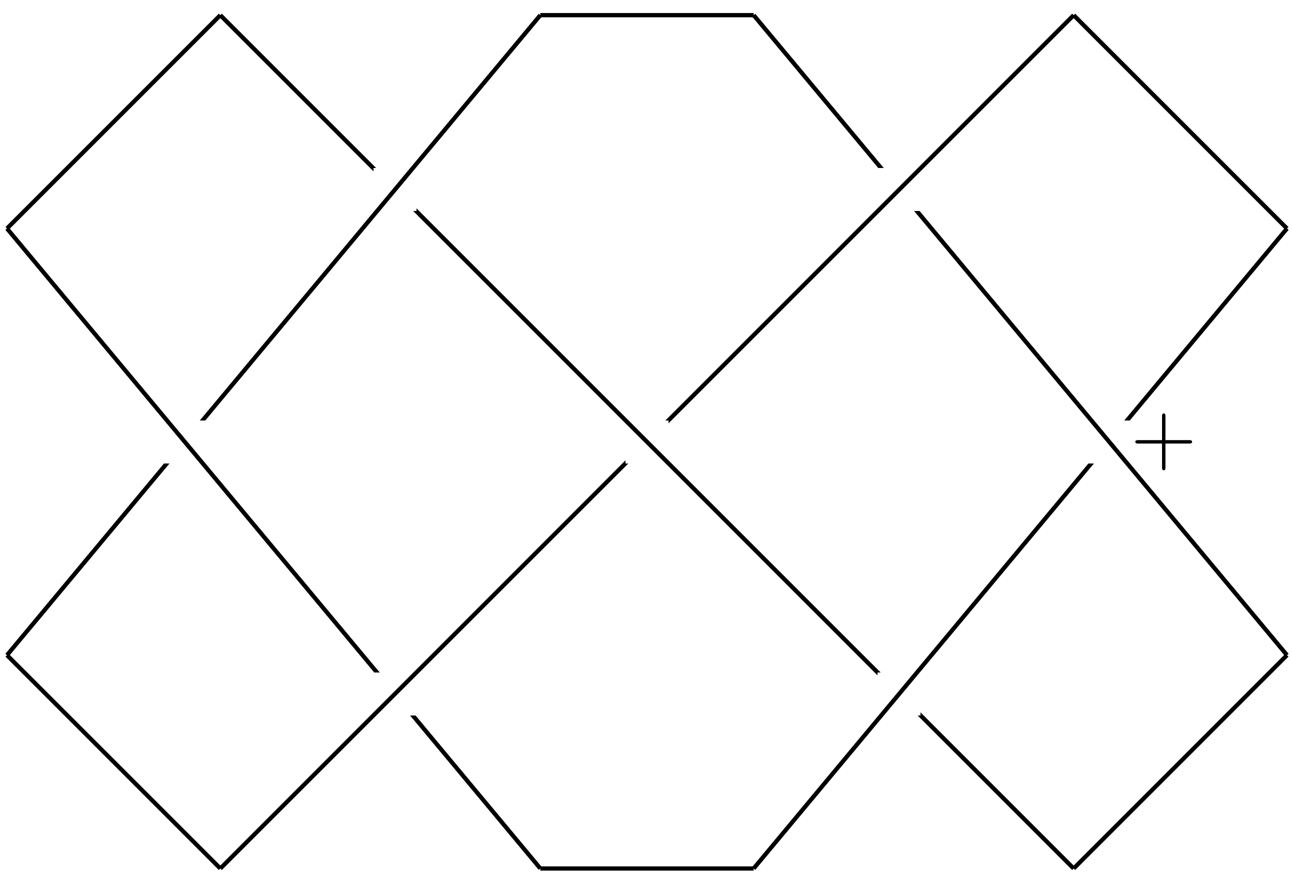
so, $P(K) = \ell^{-4} + \ell^{-2}m^2 - \ell^{-4}m^2 + \big(\ell^{-3} - \ell^{-5}\big)mH$.

Now the +Hopf link has polynomial $H = -\ell^{-1}m + \ell^{-3}m^{-1} + \ell^{-1}m^{-1}$ so

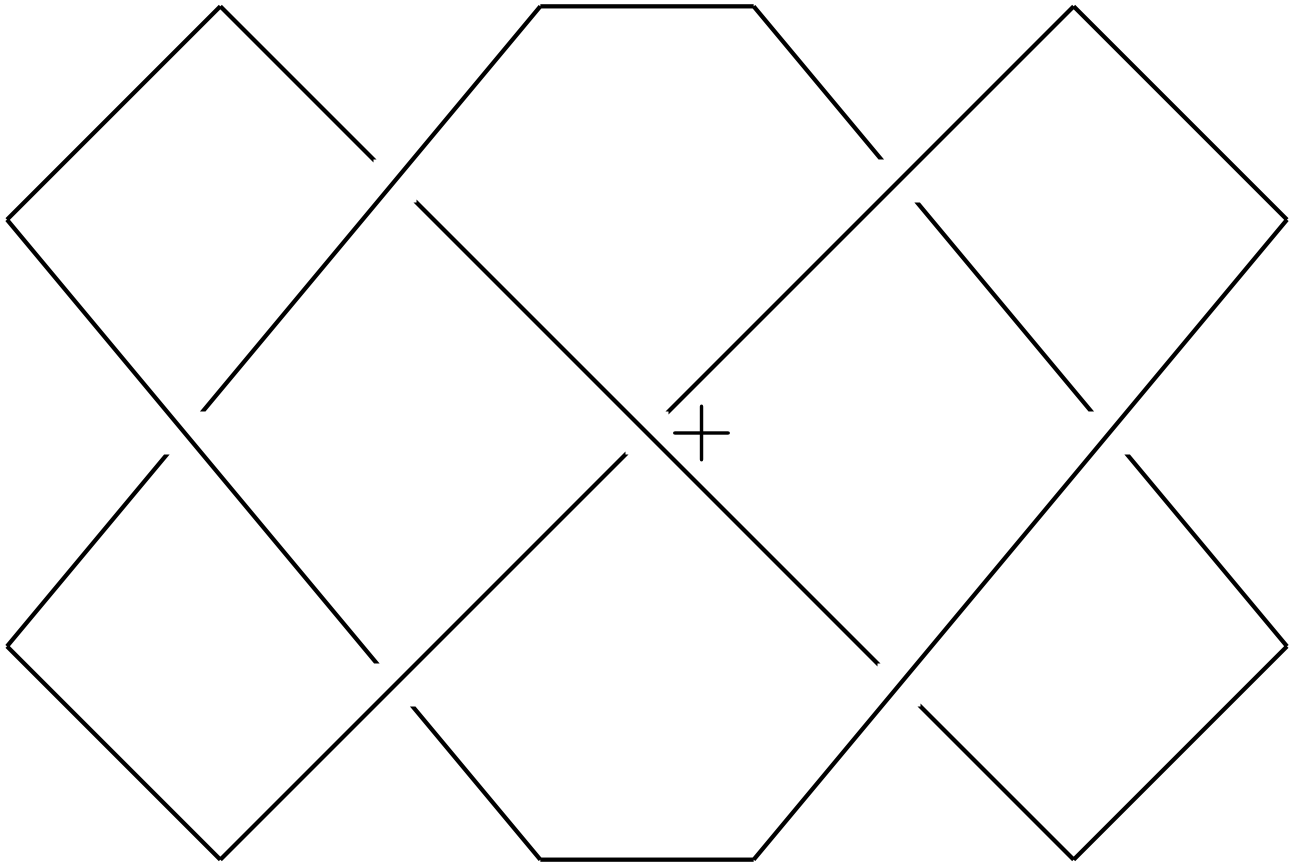
$$\begin{aligned} P(K) &= \ell^{-4} + \ell^{-2}m^2 - \ell^{-4}m^2 + \big(\ell^{-3} - \ell^{-5}\big)m(-\ell^{-1}m + \ell^{-3}m^{-1} + \ell^{-1}m^{-1}) \\ &= \ell^{-4} + \ell^{-2}m^2 - \ell^{-4}m^2 - \ell^{-4}m^2 + \ell^{-6} + \ell^{-4} + \ell^{-6}m^2 - \ell^{-8} - \ell^{-6} \end{aligned}$$

or

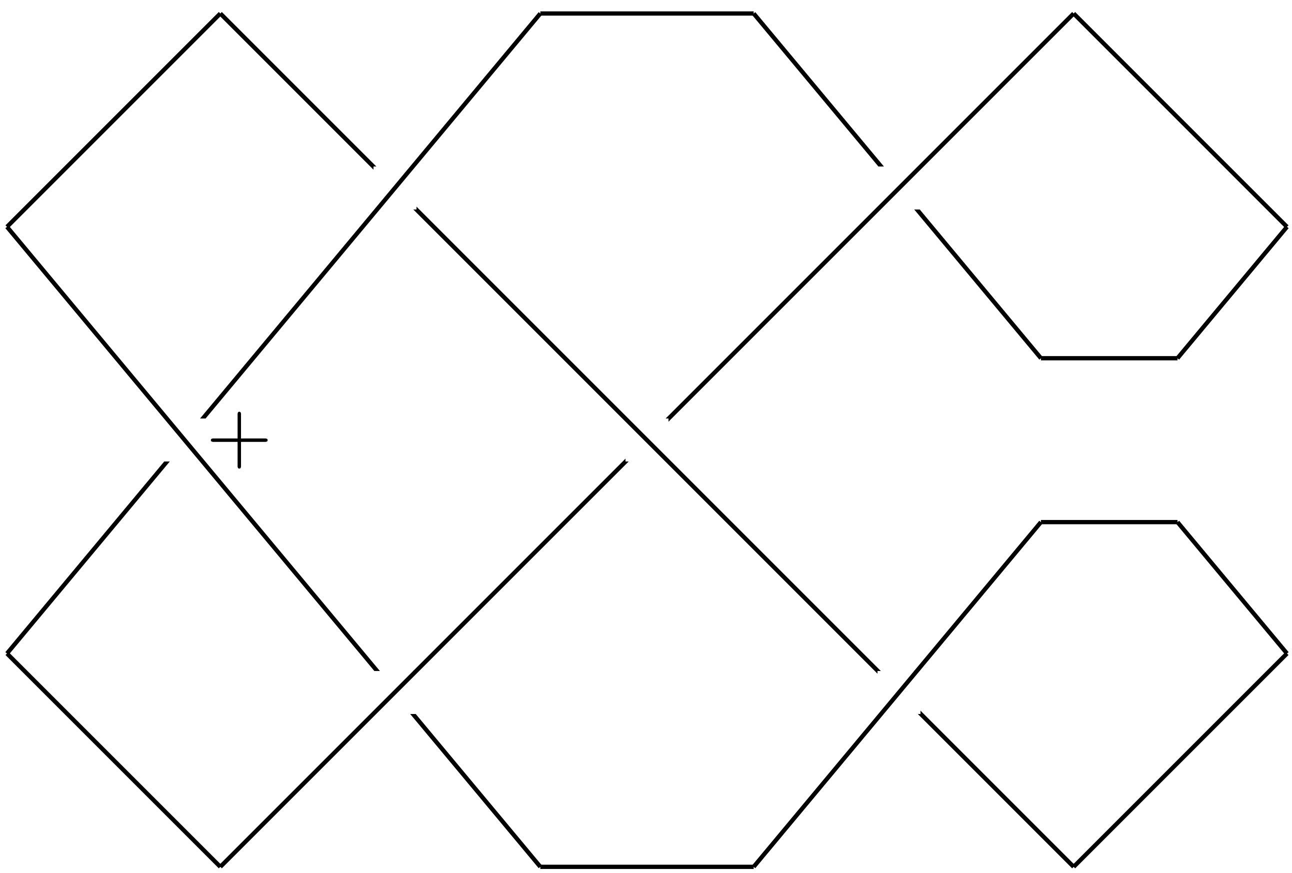
$$P(K) = -\ell^{-8} + 2\ell^{-4} - 2\ell^{-4}m^2 + \ell^{-6}m^2 + \ell^{-2}m^2\ .$$



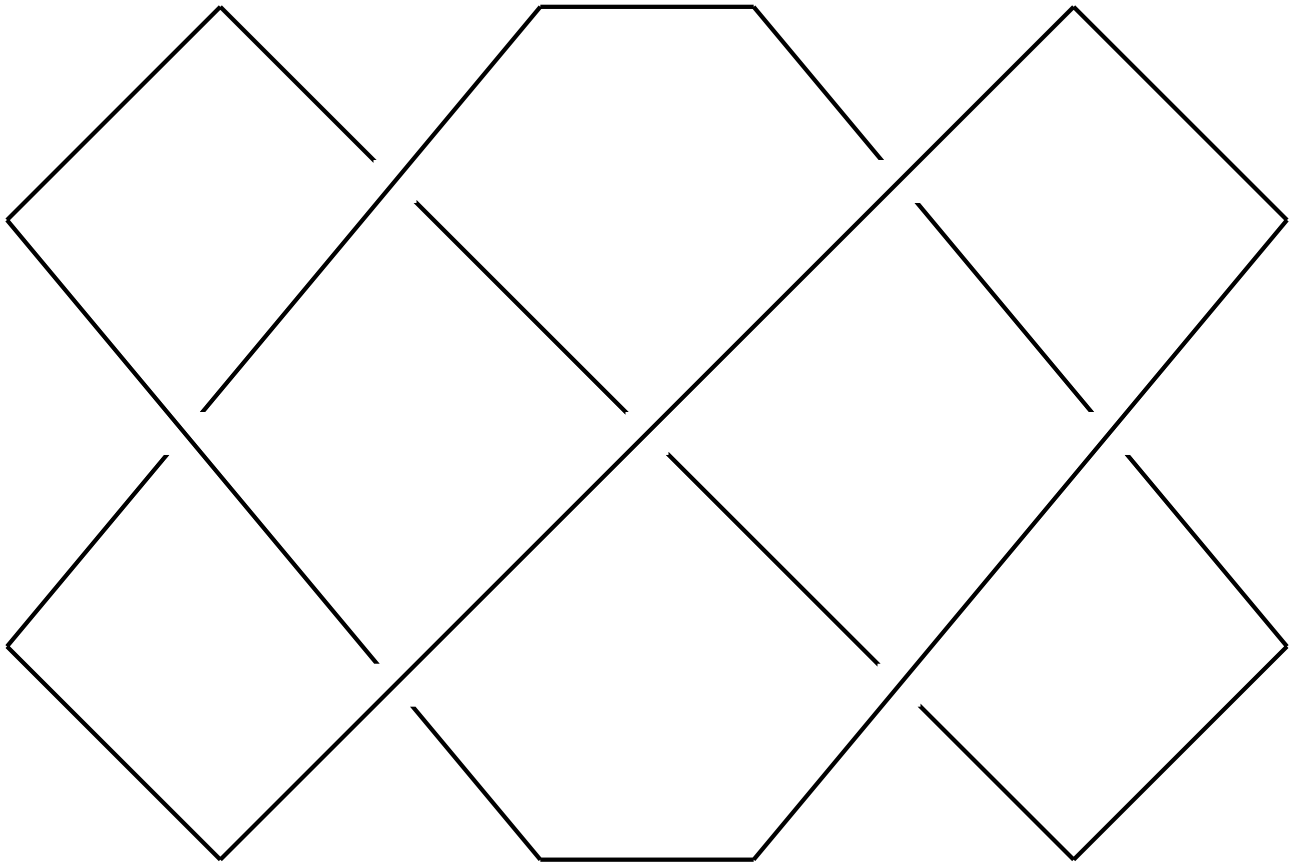
K



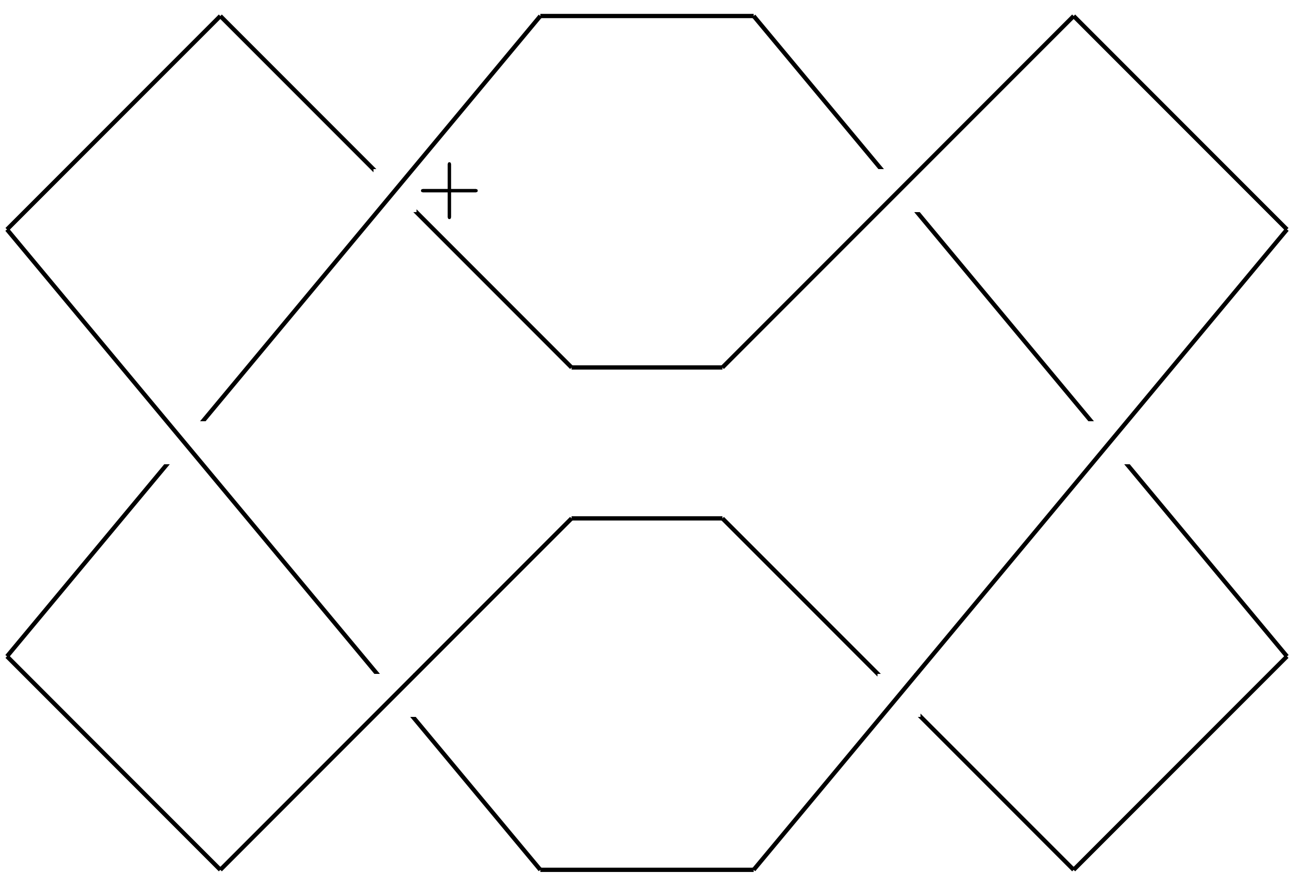
K1



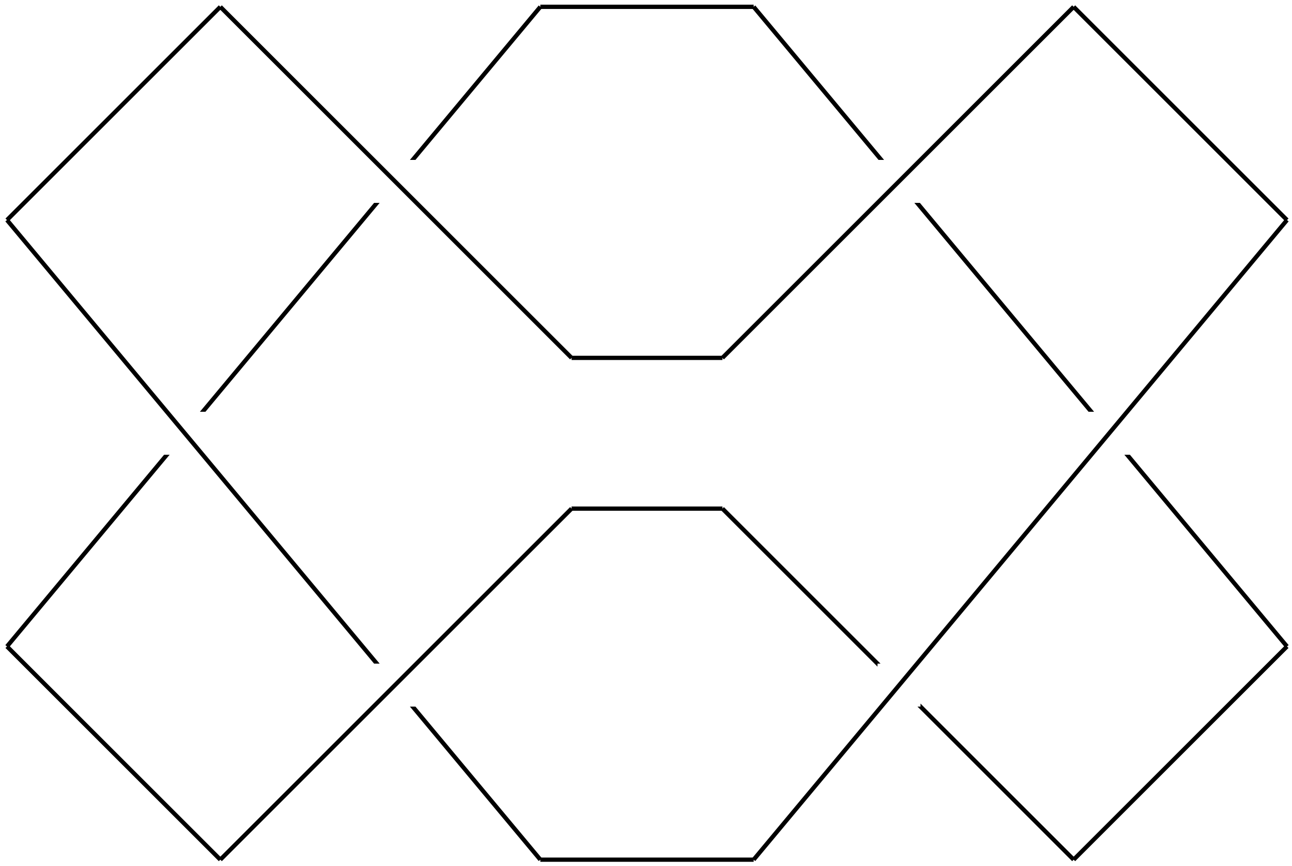
K0



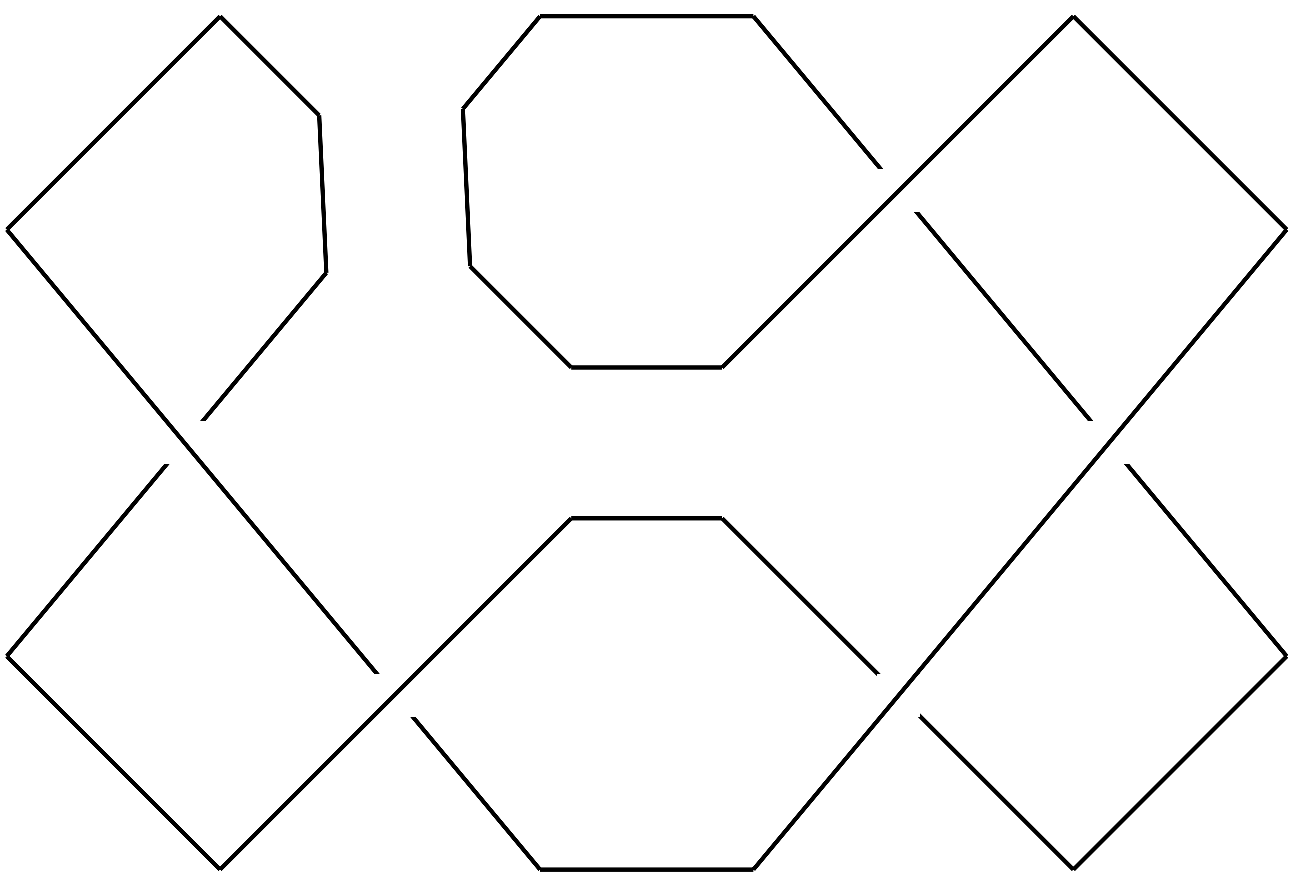
K11 unknot



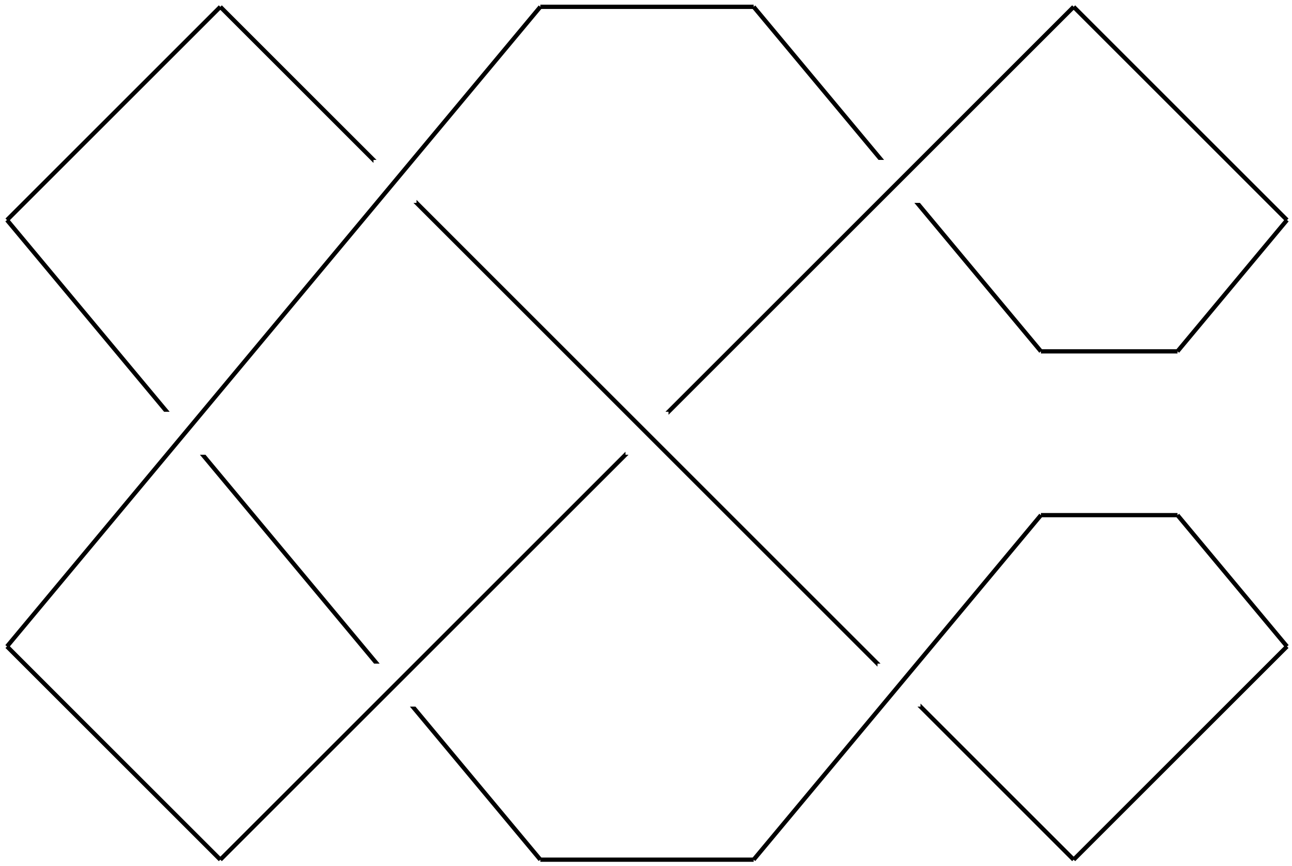

K10



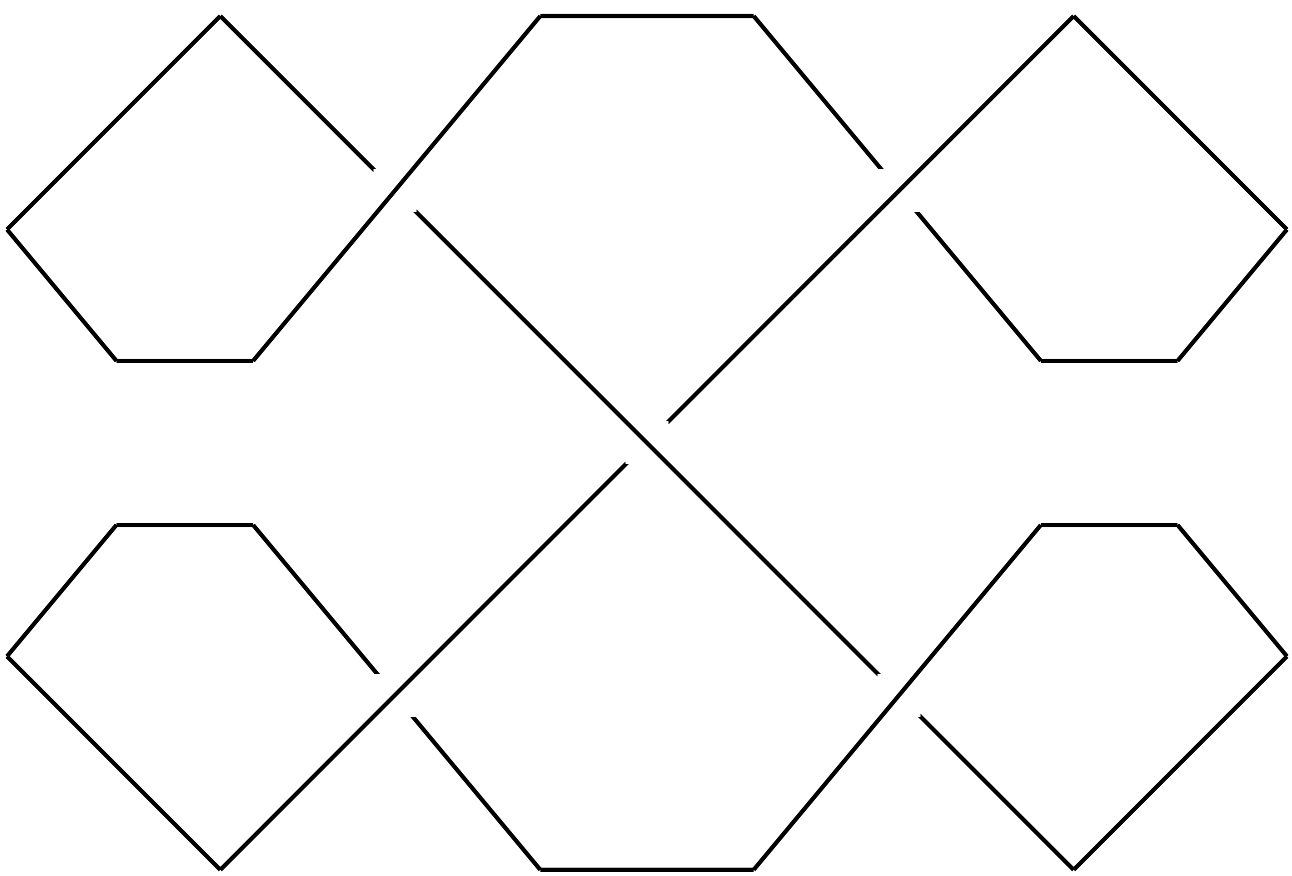
K101 +Hopf link



K100 unknot



K01 +Hopf link



K00 unknot

There are a number of "sanity checks" you can do on your HOMFLY polynomials. First of all, are all exponents even (if you have a knot or link with an odd number of components) or odd (if you have a link with an even number of components)? Second, is $P(\ell, \ell + \ell^{-1}) = \pm 1$? Or even $P(1,2) = \pm 1$? Third, do all your polynomials have lowest degree in $m$ term 0 for a knot, $m^{-1}$ for a two component link, etc.? Etc.

5. This problem is a variation on one of the problems in assignment 5. First the warm up exercise. We are given an embedding $h\colon K_8 \to \mathbf{R}^3$ which we are TOLD is achiral. This means that there is an automorphism of the graph $\theta$ and an orientation reversing homeomorphism of $\mathbf{R}^3$, $f$, such that $f \circ h = h \circ \theta$.

   There are 8 different copies of $K_7$ in $K_8$ (any set of seven vertices generates a subgraph of $K_8$ which is a $K_7$). If $\theta$ fixed at least one vertex, then $\theta$ would leave at least one $K_7$ invariant. The composition $h'\colon K_7 \subset K_8 \to \mathbf{R}^3$ given by restricting $h$ would satisfy $f \circ h' = h' \circ \theta$ and so this embedding of $K_7$ would be achiral. But $K_7$ is intrinsically chiral so this can not happen.

   What must be happening is that $\theta$ takes each embedding of $K_7$ to a different one which is the mirror image of the original. A bit more precisely, if we consider how $\theta$ acts on the set of $K_7$'s in $K_8$, we see that the orbits must each have an even number of elements since each $K_7$ can be paired with its mirror image. (This of course requires $\theta$ to have even order.)

   By the warm up exercise, $\theta$ can not fix a vertex, so it further follows that $\theta$ moves all the vertices: acting on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ $\theta$ has no one–element orbits. This also follows from the last paragraph since the action of $\theta$ on the set of $K_7$'s and the action of $\theta$ on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are isomorphic: if you know where one vertex goes you know how to map the vertices of the complementary $K_7$. Hence $\theta$ has no one–element orbits because all orbits have an even number of elements. Of course this says more since it says that $\theta$ acting on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ can not have any orbits with an odd number of elements. As an example $\theta$ can not be $(123)(456)(78)$ which fixes no element, has order 6, but has some orbits with 3 elements in them.