Here is a useful result which converts movies of the sort we can see to the ambient isotopies required by our definitions.

A *box* in $\mathbf{R}^3$ is any subset of the form $[a,b]\times[c,d]\times[e,f]$. The subset really does look like a box each of whose sides is parallel to one of the coordinate planes. Let $X\subset\mathbf{R}^3$ be a compact set (that is X is contained in some box). Let $X\subset U\subset\mathbf{R}^3$ with U an open subset of $\mathbf{R}^3$. Let $h\colon U\times[0,1]\to\mathbf{R}^3$ be an isotopy of the inclusion $U\subset\mathbf{R}^3$. Recall that this is a smooth function which is an embedding for each $t\in[0,1]$ and h_0 is the inclusion.

Thom Isotopy Extension Theorem: *Given an isotopy as above, there exists an ambient isotopy $H\colon\mathbf{R}^3\times[0,1]\to\mathbf{R}^3$ with $H(h(x,0),t)=h(x,t)$ for all $x\in X$.*

Roughly speaking, this says that if you see a motion of a set X in which you are interested and if you can extend the motion to drag along a little neighborhood of X, then you could actually find an ambient isotopy which produces the motion.

We finally return to the proof of Theorem 3. As we mention at the time, we will actually give an unlinking algorithm. Additionally we give an algorithm in which we switch only crossings between different components of L and change L to a new link $K_1\amalg\cdots\amalg K_n$ where the K_i are the subknots of L.

We begin with a more convenient description of $L_1\amalg L_2$ and $L_1\#L_2$. Any link is a compact set so we can find a box S with $L\subset S$. By making the box a little bigger, we will be able to find an open set U with $X\subset U\subset S$. Hence motions obtained by dragging links around inside boxes give equivalent links.

The disjoint union can now be succinctly described. Put $L_1\subset\mathbf{R}^3$ in a box; put $L_2\subset\mathbf{R}^3$ in a box; move the two boxes so as to be disjoint. You can find motions to see that the actual boxes you used and where you dragged them in $\mathbf{R}^3$ doesn't matter. If there is an ambient isotopy from L_1 to L_1', the motion of L_1 given by the ambient isotopy can be contained in a box and indeed, the box can be chosen to contain a neighborhood of the link under the motion. This shows that the equivalence class of $L_1\amalg L_2$ only depends on the equivalence classes of the L_1 and the L_2.

As usual, $L_1\#L_2$ is a bit more involved. Put $L_1\subset\mathbf{R}^3$ in a box and pick a point $p_1\in L_1$. A *band* is a smooth embedding $b\colon[-1,1]\times[0,1]\to\mathbf{R}^3$. Add a band to L_1 as follows. Pick an $\epsilon_1>0$ and an embedding $\vec{r}\colon[-\epsilon_1,\epsilon_1]\to\mathbf{R}^3$ which gives an arc length parameterization of the segment on which p_1 lies and so that $\vec{r}(0)=p_1$ and motion in the positive s direction traverses the curve in the same direction as the orientation. Extend this function to a band $b_1\colon[-1,1]\times[0,1]\to\mathbf{R}^3$ so that $b_1(s,0)=\vec{r}(\epsilon_1\cdot s)$ and $b_1(s,t)$ is a straight line segment lying on one side of the box containing L_1. Further require that a small neighborhood of the end of the band is perpendicular to the wall at which it ends. Further choose the embedding b_1 so that it misses L_1 except along $[-1,1]\times 0$ and the image of b_1 remains inside the box. Use b_1 to denote both the map and its image.

It is a theorem of differential topology that there is an ambient isotopy which gives the following motion of $L_1\cup b_1$. Under the motion, L_1 does not move and b_1 is pulled back towards $[-1,1]\times 0$. Given any $\delta>0$ there is a motion so that at time t the embedding is $b_1\bigl(s,1+t(\delta-1)\bigr)$. Finally, given any two bands, b_1 and b_1' and any neighborhood U of L_1, there is a $\delta>0$ and an ambient isotopy that leaves L_1 fixed, shrinks the band b_1 using δ so that the shrunken band lies inside U, does a further motion inside U so that band is now b_1' shrunken using δ, and finally expands the shrunken band back out to b_1'. The entire motion takes place inside the box.

Do the same thing to L_2 to get a band b_2. Move the boxes for L_1 and L_2 until they are disjoint. Add one more band, b_3 such that

1) $b_3(s, 0)$ is the line segment at which b_1 ends; the direction of increasing s on b_3 agrees with the direction of increasing s on b_1; and b_3 is initially perpendicular to the wall at which it begins.
2) $b_3(s, 1)$ is the line segment at which b_2 ends; the direction of increasing s on b_3 agrees with the direction of increasing s on b_2; and b_3 is finally perpendicular to the wall at which it ends.
3) For all $0 < t < 1$ and $-1 \leq s \leq 1$, $b_3(s, t)$ lies in $\mathbf{R}^3$ minus the two boxes.

Glue b_1 to b_3 to b_2 to get a band b. The link $L_1 \# L_2$ is formed from $(L_1 \amalg L_2) \cup b$ by deleting the image of $(-1, 1) \times [0, 1]$. With a bit of energy, one can see that its equivalence class only depends on the equivalence classes of L_1 and L_2, together with the components of the links in which the points p_1 and p_2 lie.

One reason for being careful with the connected sum is that there is a generalization, called the *band connected sum* and often written $L_1 \#_b L_2$, which is a generalization of the connected sum. Informally, start with $L_1 \amalg L_2$ and start a band at p_1 (just as for the connected sum) and drag the band anywhere you like in $\mathbf{R}^3 - (L_1 \amalg L_2)$ before attaching it at p_2.

The band connected sum is a generalization of the connected sum (which severely restricts how the band can be placed). It is not well–defined by just specifying the components at which the band begins and ends. The band sum of even two unknots can be a non–trivial knot with a non–trivial HOMFLY polynomial. Hence one can not even give a formula for the HOMFLY polynomial of a band sum it terms of the pieces. The key thing that is special in the connected sum case is that the final link does not depend on how many twists are in the band, where as this is false for a general band sum. You can begin by untwisting the band all right, but if the band passes through L_2, when you twist L_2 the band will change.

We now begin to describe our algorithms. Both algorithms begin by numbering the components of the link. Let K_i be the i^{th} subknot in this labelling and picking a point $p_i \in K_i$ for $1 \leq i \leq n$ where n is the number of components. Orient each component as well.

Unsplitting Algorithm: *Begin at p_1 and traverse K_1 in the direction determined by the orientation. When we come to a crossing between K_1 and any other crossing, switch it if necessary to insure that it is an under crossing. When we return to p_1 move on to K_2. In general, we start at p_i and traverse K_i in the direction determined by the orientation. If we encounter a crossing whose other strand comes from K_j with $j > i$, switch if necessary so that the crossing is an under crossing. When we have traversed K_1 through K_{n-1}, the resulting link is $K_1 \amalg \cdots \amalg K_n$.*

Remark: A second description of this process is to just grap the first component and pull it down, switching any crossings that try to hang you up. This *may* explain how some unlinking chemical processes work.

Proof: Write L_1 for the sublink of L consisting of all the components *except* K_1. Let $L^{(1)}$ be the link we obtain after traversing K_1. Both K_1 and L_1 are sublinks of $L^{(1)}$ since L and $L^{(1)}$ only differ, if at all, in the crossings between K_1 and L_1. In $L^{(1)}$ all the crossings of K_1 are below L_1 by construction so we can drop K_1 straight down until it fits in a box and L_1 fits in a disjoint box. Hence $L^{(1)} = K_1 \amalg L_1$.

Number the component of L_1 by subtracting one from the number they acquire as a sublink of L. The second step of algorithm applied to L is just the first step of the algorithm applied to L_1 and we are done. ∎

Unknoting Algorithm: *Let K be an oriented knot and choose a point on it. Traverse the knot starting at p in the direction of the orientation. When you come to a crossing, if it is the first time you have traversed it, switch it if necessary to insure that it is an undercrossing. (If this is the second time you have seen it, it will be an over crossing.) When you have traversed the knot once, the new knot is the unknot.*

Remarks: You can put the two algorithms together to unlink a link. Just change the crossings so that "first time is an undercrossing". When the two strands of a crossing come from different components, this is determined by the way the components were numbered.

Proof: Let K be a regular projection of a knot satisfying "first crossing under" starting at $p_0 \in K$. It suffices to prove K is a regular projection of the unknot. If there are no crossings this is easy, so assume we have some crossings. The first crossing is an under crossing and as we continue around the knot we may encounter several under crossings in a row, but eventually we will hit our first overcrossing. Pick $p_1 \in K$ after the last under crossing and before the first over crossing. After p_1 we see a sequence of one or more over crossings and we pick $p_2 \in K$ after the last over crossing and before the next under crossing. If we have been around the knot completely, pick $p_2 = p_0$. Continue around the knot picking points $p_i \in K$ where p_{2i+1} is a point between an under crossing and an adjacent over crossing while p_{2i} is a point between an over crossing and an adjacent under crossing. Between p_i and p_{i+1} we see only crossings of the same type. Eventually you will get around the knot and encounter the initial under crossing and you are done. Note the interval between p_{2n-1} and p_0 consists of a sequence of over crossings for some n. Consider the product $\prod_{i\neq j} \big((p_i)_x - (p_j)_x\big)\big((p_i)_y - (p_j)_y\big)\big((p_i)_z - (p_j)_z\big)$. If necessary, jiggle the embedding $K \subset \mathbf{R}^3$ so that this product is not zero. Slide the embedding up if necessary to insure the knot lies above the xy–plane, which is the plane $z = 0$.

Drop the segment between p_0 and p_1 straight down to the plane $z = -2n$. This can be done because all crossings of this segment lie under the rest of the knot. The union of the projected segment plus the vertical lines from p_0 and p_1 to their projections into $z = -2n$ plus the knot minus the segment is equivalent to the original knot. Next push the segment from p_1 to p_2 down to the plane $z = -2n + 1$. This can be done since the only crossings below this strand have been pushed all the way down to $-2n$. This knot is still equivalent to the original one. Continue in this fashion around the knot.

The projection into the xy–plane is no longer regular because of the vertical drops at the p_i but the image in the xy–plane has not changed. Each of the segments between a pair of the p_i now lies in a plane for which z is a constant and each segment lies in a different plane.

Given an embedded arc in a plane between two points, there is an ambient isotopy of the plane starting with the arc and finishing with the straight line between the points. This is not trivial to prove but seems reasonable. Imagine the arc is made of a very stretched piece of rubber band so when you stop holding it down, except at the end points, it will snap taut and that will be a straight line. Hence the original knot is equivalent to one with the segments in different planes parallel to the xy–plane and with the arcs in each plane being straight lines.

Look at a projection into the yz–plane. Because the coordinates of the p_i are all distinct, this projection has no crossings. Hence it is regular and the knot is the unknot. ∎