Homework Problems (March 30, 1998)

Mathematics 522— Stochastic Differential Equations
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Test 2 will be in class on Wednesday, April 8

A postscript file of the most recent version of this sheet will always be in the class folder: `/afs/nd.edu/coursesp.98`

Please see me if there are concepts that you do not feel comfortable with.

1 Problems due Monday, April 6, 1998

Let X be a random variable with normal probability distribution function having mean μ and variance σ^2. Show that the moment generating function $M_X(t) := E(e^{tX})$ of X is $e^{\mu t + \frac{\sigma^2 t^2}{2}}$. Assuming that $\mu = 0$ compute $E(X^n)$. Deduce from problem (**??**) that any finite linear combination of independent normal variables is normal (you can use the fact that the pdf of a random variable X is determined by its moment generating function $M_X(t)$ if $M_X(t)$ exists in a neighborhood of $t = 0$). Let B_t denote the standard Brownian motion on $(C[0, \infty),)$ with $B_0 = 0$ and the measure algebra $:= \sigma(B_s; s \in [0, \infty))$. Deduce from problem (**??**) that for any $\alpha > 0$ the $e^{\alpha B_t - \frac{\alpha^2 t}{2}}$ is a martingale relative to the filtration of given $_t := \sigma(B_s; s \leq t)$. Let $f(x)$ be a uniformly continuous real valued function on $\cap[0, 1]$ relative to the usual metric on $[0, 1]$. Show that f has a unique extension to a continuous function on $[0, 1]$.

2 Problem due Monday, March 16, 1998

Let $X_1, X_2, \ldots$ be identically distributed and independent random variables satisfying $E(|X_i|) < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Show that if $L := \lim_{n\to\infty} S_n/n$ exists then

$$L = \lim\ \sup_{n\to\infty} \frac{X_{k+1} + \cdots + X_{k+n}}{n}.$$

3 Problems due Monday, February 23, 1998

E10.4, E10.7 (For 10.7 show only that N_n is a martingale and using this compute what $E(T)$ is: you will need to use the first part of the problem done in class.)

4 Problems due Monday, February 16, 1998

E10.1, E10.3

5 Problems due Monday, February 9, 1998

Do *Problems* E9.2

6 Problems due Monday, February 2, 1998

Do *Problem* E5.1

7 Problems due Wednesday, January 28, 1998

Let $C[0,1]$ denote the space of continuous real valued functions on the unit interval, $[0,1]$. We define the sup-norm for $f \in C[0,1]$ by $||f|| := \max_{x\in[0,1]} |f(x)|$. We define the distance between two functions $f, g \in C[0,1]$ by $d(f,g) := ||f-g||$. For $x \in [0,1]$, we denote the evaluation map $_x : C[0,1] \to$ by $_x(f) := f(x)$. Given $x \in [0,1]$ prove that $_x$ is a continuous mapping from $C[0,1]$ with the topology given by the using the above distance and with its usual metric topology.

We let $(C[0,1])$ denote the σ-algebra associated $C[0,1]$ with the toplogy given by the metric above— unless it is said otherwise this will always be the toplogy referred to when I talk about open sets of $C[0,1]$. We let $(\{_x | x \in [0,1]\})$ denote the σ-algebra generated by all of the evaluation maps for points $x \in [0,1]$. Prove that $(C[0,1]) = (\{_x | x \in [0,1]\})$. Let me give a suggested solution for the last problem— justifications of the steps are needed. Note that since $_x$ is continuous by Problem 1, we have that $^{-1}_x(a,b)$ is open in $C[0,1]$. Therefore (why?) $(\{_x | x \in [0,1]\} \subset (C[0,1])$. To show the other inclusion prove first that we can use the sets $B_r(f) := \{g \in C[0,1] | d(g,f) \leq r\}$ parameterized by $f \in C[0,1]$ and $r \in, r > 0$ as a generating set for $(C[0,1])$. To show this you should note that since any open set of $C[0,1]$ is a countable union of the open balls $B_r(f) := \{g \in C[0,1] | d(g,f) < r\}$, it follows (why?) that $(C[0,1])$ is generated by these open balls. Since $B_r(f) = \cup_{q\in, 0<q<r} B_q(f)$ and $B_r(f) = \cap_{q\in, q>r} B_q(f)$ it follows (why?) that the closed balls $B_r(f)$ also generate $(C[0,1])$. Therefore if we show that the sets $B_r(f) \subset (\{_x | x \in [0,1]\})$ we will have that $(C[0,1]) \subset (\{_x | x \in [0,1]\})$ and therefore that $(C[0,1]) = (\{_x | x \in [0,1]\})$. Why is it true that the set $\cap_{q_1\in, q_1>r} \left(\cap^{-1}_{q_2\in(\cap[0,1])} (f(q_2) - q_1, f(q_2) + q_1) \right)$ is

1. an element of $(\{_x | x \in [0,1]\})$;
2. which is in fact equal to $B_r(f)$.