Mathematics 522: Test I

February 25, 1998

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Name: ....

Do Problems 6 and your choice of three of the remaining five problems. There are blank pages at the end if you need them.

Let $(\Omega, , P)$ be a probability triple.

1. State the Borel-Cantelli Lemmas; and

2. State one of the Fatou Lemmas.

Let $(\Omega, , P)$ be a probability triple. Prove one of the Borel-Cantelli Lemmas or one of the Fatou Lemmas.

Let $X_1, X_2, \ldots$ be a sequence of real valued random variables on a probability triple $(\Omega, , P)$. Show that $P(\{X_n \text{ converges}\}) = 0$ or $= 1$. *(HINT: Point out why* $\{X_n \text{ converges}\}$ *is a tail event, and quote an appropriate result about tail algebras.)*

Let $(\Omega, , \{_n\}, P)$ be a filtered space. Let $X \in L^1(\Omega, , P)$.

1. Give the definition of a martingale on $(\Omega, , \{_n\}, P)$.

2. Letting $X_n := E(X|_n)$ for $n = 0, 1, \ldots$, prove that $X_0, X_1, \ldots$ is a martingale.

Let $(\Omega, , \{_n\}, P)$ be a filtered space. Let $X_0, X_1, \ldots$ be real valued functions $\in L^1(\Omega, , P)$. Assume that the process $X_0, X_1, \ldots$ is adapted to the filtration $\{_n\}$.

1. What does it mean when we say that the stochastic process $X_0, X_1, \ldots$ is adapted to the filtration $\{_n\}$?

2. Assume that $\lim_{i \to \infty} E(|X_i - 2|) = 0$, where 2 denotes the constant function. Let $T : \Omega \to^+$ denote the map which sends $\omega \in \Omega$ to the smallest integer n such that $X_n(\omega) \in [1, 4]$ or to ∞ if no $X_n(\omega) \in [1, 4]$. Show that T is an almost surely finite stopping time.

3. Show that T is in fact almost surely bounded, i.e., that there exists a real number $K > 0$ such that $T \leq K$ off of a set of P-measure zero.

Let $X_0, X_1, \ldots$ denote a martingale on a filtered space $(\Omega, , \{_n\}, P)$. Let T denote a stopping time on $(\Omega, , \{_n\}, P)$.

1. Give the definition of the stopped process X^T; and

2. Show that X^T is a martingale.