Math 601 Midterm
October 9, 2000

Instructions: Please do all the questions. Please do not hesitate to ask for clarification. You may use the textbook and your notes for reference.

Good luck!

1. Suppose that G is a finite group and H is a normal subgroup of G so that $[G : H]$ is odd. Suppose further G has a subgroup K of order 8. Show that $K \subseteq H$.

2. Let G be a group. Let $S = \{ghg^{-1}h^{-1} : g, h \in G\}$. Let $C =< S >$.
   (C is called the **commutator** subgroup of G.)
   (a) Show that $C = \{s_1 s_2 \ldots s_n : s_1, \ldots, s_n \in S\}$.
   (b) Show that C is a normal subgroup of G.
   (Suggestion: start by considering each element of S, then use (a).)
   (c) Show that G/C is abelian.
   (d) Suppose that $\Phi : G \to A$ is a homomorphism between G and an abelian group A. Let $\pi : G \to G/C$ be the canonical projection. Show that there is a unique homomorphism $\tilde{\Phi} : G/C \to A$ so that $\Phi = \tilde{\Phi} \circ \pi$:

3. Let $n \in$. Let G be, as a set (but *not* as a group), the Cartesian product $/2 \times /n$.
   Define a binary operation on G by
   $$(\bar{i}, \bar{j})(\bar{k}, \bar{m}) = (\bar{i} + \bar{k}, (-1)^k \bar{j} + \bar{m}).$$
   (a) Show that G is a group.
   (b) Verify that G is an internal semidirect product of its cyclic subgroups $/2 \times \{\bar{0}\}$ and $\{\bar{0}\} \times /n$.
   (Recall that G is an internal semidirect product of two subgroups H and N if N is normal, $G = HN$ and $H \cap N = \{e\}$.)
   (c) Do you recognise this group?

4. Let H be a group.
   (a) Show that there is a category $\mathcal{C}$ in which:
   - an object is a group containing H as a subgroup, and
   - a morphism is a group homomorphism that restricts to the identity on H.

   (b) Show that for each object G of $\mathcal{C}$ there is a unique morphism $H \to G$.
   (So H is an **initial** (or **universal** or **universally repelling**) object in the category.)
   (c) How would (a) and/or (b) change if we sought a category in which an object is a group having a subgroup isomorphic to H?