Final Exam
Math 601
10 a.m. December 14
Due: 12 noon December 15, 2000

Instructions: Please do questions 1 through 3. You may obtain extra points by doing question 4. To maximize credit, please present your answers in a clear and precise form. Remember: it will be easier for me to understand your work if I can **read** it!

Good luck!

1. Let G be a group of order 231. Let H be a Sylow 11-subgroup, K a Sylow 7-subgroup, and L a Sylow 3-subgroup of G.
   (a) Show that H and K are both normal subgroups of G.
   (b) Show that G has a cyclic subgroup of order 77.
   (c) Show that $G = HKL$.
2. Consider / as a group under addition.
   (a) Prove that every finitely generated subgroup of / is cyclic.
   (b) Prove that / is not a finitely generated group.
3. Let R be a commutative ring with identity element $1 \neq 0$. Recall that an element $r \in R$ is *nilpotent* if there exists $n \in \mathbf{N}$ such that $r^n = 0$.
   (a) Show that the set of nilpotent elements of R is an ideal.
   (b) Show that if r is nilpotent then $1 - r$ is a unit.
   (c) Suppose $a_0 \in R$ is a unit, and $a_1, \ldots, a_n$ are nilpotent. Let $f = a_0 + a_1 x + \ldots + a_n x^n \in R[x]$. Show that f is a unit in $R[x]$.

(Bonus) Let $\mathcal{C}$ be a concrete category. Suppose that for any two objects A, B of $\mathcal{C}$, there is an object $A * B$ of $\mathcal{C}$ and morphisms

$$\iota_A : A \to A * B, \ \iota_B : B \to A * B$$

so that $A * B$, together with the morphisms ι_A and ι_B, is a coproduct of A and B.

(a) Suppose that $A,\ B,\ C$ are three objects of $\mathcal{C}$. Show that $(A * B) * C$ (with appropriate morphisms) is a coproduct of $A,\ B,\ C$.
(b) Show that $A * (B * C)$ is a coproduct of $A,\ B,\ C$.
(c) Conclude that $(A * B) * C$ is isomorphic to $A * (B * C)$.
(d) Did we need the hypothesis that $\mathcal{C}$ is a concrete category?
(Suggestion: draw the diagrams.)