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Mathematics 603: Real Analysis
Fall Semester 2000
Final Exam
December 14, 2000

For a proof, state precisely the theorem(s) that you used.

1. (10 points) State PRECISELY Lebesgue's Dominated Convergence Theorem.

2. (10 points) State PRECISELY Fubini's Theorem on integrals.

3. (10 points) State PRECISELY Lebesgues's Differentiation Theorem.

4. (21 points) Give the PRECISE definition of the following

(a) A measurable function.

(b) The convergence of f_k to f in measure as $k \to \infty$.

(c) An absolute continuous function defined on $[a, b]$.

5. (49 points) True or False questions.

(a) Let f and f_k be measurable and a.e. finite in E, where E is a measurable set. If $f_k \to f$ a.e. on E, then f_k also converges to f in measure.

(b) Let f and f_k be measurable and a.e. finite in E, where E is a measurable set. If f_k converges to f in measure, then there is a subsequence f_{k_j} which converges to f almost everywhere.

(c) Let $f_k \geq 0$ be measurable and a.e. finite in E, where E is a measurable set. Then we always have

$$\int_E \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int_E f_k.$$

(d) Let f_k be measurable and a.e. finite in E, where E is a measurable set. If $f_k \nearrow f$, then we always have

$$\int_E f_k \to \int_E f.$$

(e) If E is measurable, then we always have

$$\lim_{Q\to x} \frac{|Q \cap E|}{|Q|} = 1 \text{ for a.e. } x \in E,$$

where Q denotes the cubes centered at x.

(f) If f is a finite monotone increasing function defined on $[a, b]$, then f' exists almost everywhere, and $\int_a^b f' \leq f(b-) - f(a+)$.

(g) If $f \in L(R^n)$, then the Hardy-Littlewood maximal function f^* is always in weak-$L(R^n)$, and there exists a constant c which depends only on n, such that for all $\alpha > 0$,

$$|\{x \in R^n;\ f^*(x) > \alpha\}| \leq \frac{c}{\alpha} \int_{R^n} |f|.$$

6. (10 points) If $f \in L[0,1]$, show that $x^k f(x) \in L[0,1]$ for $k = 1, 2, 3, \cdots$ and $\int_0^1 x^k f(x)dx \to 0$.

7. (10 points) If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

8. (10 points) Let f_k be a sequence of measurable functions defined on a measurable set E with $|E| < \infty$. Assume that for every $x \in E$, $\sup_{1 \leq k < \infty} |f_k(x)| = M_x < \infty$. Show that for any $\varepsilon > 0$, there exists a closed $F \subset E$ and a finite M such that $|E \setminus F| < \varepsilon$, and

$$|f_k(x)| \leq M \text{ for all } 1 \leq k < \infty \text{ and all } x \in F.$$

9. (10 points) Let E be a measurable subset of R^2 such that for almost every $x \in R^1$, $\{y;\ (x, y) \in E\}$ has R^1-measure zero. Show that $|E| = 0$, and that for almost every $y \in R^1$, $\{x;\ (x, y) \in E\}$ has R^1-measure zero.

10. (10 points) If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \le V[f; a, b].$$

(Hint: If $V(x) = V[f; a, x]$, what is $V'(x)$?)