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Mathematics 603: Real Analysis
Fall Semester 2000
Midterm Exam
October 11, 2000

For a proof, state precisely the theorem(s) that you used.

1. (10 points) State PRECISELY the Mean Value theorem for Riemann-Stieltjes integral.

2. (10 points) State PRECISELY **two** equivalent definition of a Lebesgue-measurable set.

3. (10 points) Give the PRECISE definition of the following
(a) Borel σ-algebra. (b) G_δ set.

4. (10 points) If $\{E_k\}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ has measure zero.

5. (30 points) True or False questions.

(a) Let f be bounded and ϕ be monotone increasing on $[a, b]$. If $\sup_\Gamma L_\Gamma = \inf_\Gamma U_\Gamma$ (here L_Γ, U_Γ are the lower and upper Riemann-Stieltjes sums corresponding to the partition Γ) then the Riemann-Stieltjes integral $\int_a^b f d\phi$ always exists.

(b) A function of bounded variation on the finite interval $[0, 1]$ can always be written as the difference of two monotone increasing functions.

(c) If E is measurable, then for any set A (measurable or not)

$$|A|_e = |A \cap E|_e + |A \cap E^c|_e. \qquad (E^c \text{ is the complement of } E)$$

(d) If E_1 and E_2 are measurable, then the following is always true.

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|.$$

(e) For a sequence of sets $\{E_k\}$ (measurable or not), if $E_k \nearrow E$, then we always have $\lim_{k\to\infty} |E_k|_e = |E|_e$.

(f) For a sequence of sets $\{E_k\}$ (measurable or not), if $E_k \searrow E$ and $|E_1|_e < \infty$, then we always have $\lim_{k\to\infty} |E_k|_e = |E|_e$.

6. (10 points) Let Z be a subset of R^1 with measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero.

7. (10 points) Suppose that E_k are measurable. Prove that

$$|\liminf E_k| \leq \liminf |E_k|.$$

8. (10 points) Suppose that $E \subset [-1, 1]$ is measurable and $|E| > 1$. Let $F = \{x; \quad -x \in E\}$. Prove that the set $B = E \cap F$ has a positive Lebesgue measure.