Name: ______________________________

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MATH 603: Basic Real Analysis I – FALL 2002

FINAL EXAM

Problem	Possible points	SCORE
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
TOTAL	75	

1. Let $f : X \to \mathbb{R}$ be a measurable function.
Prove that for any $p > 0$

$$\int_X |f(x)|^p d\mu(x) = \int_0^\infty pt^{p-1}F(t)dt,$$

where $F(t) = \mu(\{x : |f(x)| > t\})$.

2. Let μ be a finite measure on X.
Prove that a non-negative function f is in $L^1(X, d\mu)$ if and only if the series

$$\sum_{n=0}^\infty 2^{-n}\mu(\{x : f(x) > 2^{-n}\})$$

converges.

3. Give an example of a sequence of functions, (f_n), Lebesgue integrable on $[0, 1]$ and such that

(i) $f_n \to f$ a.e.

(ii) $\lim_{n\to\infty} \int_{[a,b]} f_n(x)dx = \int_{[a,b]} f(x)dx$

(iii) there is no $g \in L^1([0, 1])$ such that, for $|f_n(x)| \le g(x)$ a.e. $\forall n$.

4. Let (X, M, μ) be a σ-finite measure space, N a sub-σ-algebra of M and $\nu = \mu \restriction N$.
Prove:
If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ such that

$$\int_E f d\mu = \int_E g d\nu$$

for all $E \in N$.

5. Two (signed) measures ν_1 and ν_2 are called *equivalent* if $\nu_1 \ll \nu_2$ and $\nu_2 \ll \nu_1$.

Let μ be a measure on X and f_1, f_2 two μ-integrable real functions on X. Define signed measures $d\nu_i = f_i d\mu,\ i = 1, 2$. Prove that ν_1 and ν_2 are equivalent if and only if $\mu(N_1 \Delta N_2) = 0$, where $N_i = \{x \in X; f_i(x) \neq 0\}$.

6. Let f and g be two functions absolutely continuous on $[a, b]$. Prove that the integration by parts formula holds for f and g :

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\mid_a^b - \int_a^b f'(x)$$

g(x) dx .

7. Given two compact metric spaces, X and Y, let $C(X, Y)$ be the set of all continuous mappings of X into Y. Define distance in $C(X, Y)$ by

$$\rho(f, g) = sup_{x \in X} \rho(f(x), g(x)) \ .$$

a) Prove that $C(X, Y)$ is a metric space and that it is complete.

b) Prove the following generalization of Arzela's theorem:
A set $D \subset C(X, Y)$ is pre-compact if and only if D is an *equicontinuous* family of functions, i. e. , given any $\epsilon >$ 0, there exists a number $\delta > 0$ such that $\rho(x', x'') < \delta$ implies $\rho(f(x'), f(x'')) < \epsilon$ for all $x', x'' \in X$ and all $f \in D$.