MATH 604 TAKE-HOME EXAM
Due March 15, 2004

1. Let l^∞ be the collection of bounded sequences in $\mathbb{C}$ equipped with the norm $\|x\|_\infty = \sup_k |x_k|$ for $x = (x_1, x_2, \cdots)$.

(1) Prove $(l^\infty, \|\cdot\|_\infty)$ is a Banach space.

(2) Let c_0 be the collection of sequences in $\mathbb{C}$ convergent to 0. Prove c_0 is a closed subspace of l^∞.

(3) For any $x = (x_1, x_2, \cdots) \in l^\infty$, find $\text{dist}(x, c_0)$. Can this distance be realized by an element in c_0? Justify your answers.

2. Let T be a bounded linear operator in a Hilbert space $\mathcal{H}$ and satisfy $|(Tx, x)| \geq c\|x\|^2$ for any $x \in \mathcal{H}$. Then T has a bounded inverse in $\mathcal{B}(\mathcal{H})$.

3. Let $\mathcal{X}$ be a NVS. A nonempty subset $O \subset \mathcal{X}$ is *weakly open* if for any $x_0 \in O$ there exists a finite collection of $f_1, \cdots, f_n \in \mathcal{X}^*$ such that

$$x_0 + \cap_{k=1}^n \{x \in \mathcal{X}; |f_k(x)| < 1\} \subset O.$$

A subset C is weakly closed if C^c is weakly open. Prove the following results.

(1) $\mathcal{X}$ is weakly open; a countable union of weakly open subsets is weakly open; a finite intersection of weakly open subsets is weakly open. Moreover, $\{x \in \mathcal{X}; |f(x)| < 1\}$ is weakly open for any $f \in \mathcal{X}^*$.

(2) Any weakly open subset is open (in the norm topology); any weakly closed subset is closed; the identity map $i : (\mathcal{X}, \|\cdot\|) \to (\mathcal{X}, w)$ is continuous.

(3) If $\mathcal{X}$ is infinite dimensional, any nonempty weakly open subset is unbounded. (Hint: Prove the following statement first: If $\cap_{k=1}^n \mathcal{N}(f_k) = \{0\}$ for some $f_1, \cdots, f_n \in \mathcal{X}^*$, then $\mathcal{X}$ is finite dimensional.)

(4) Closed subspaces are weakly closed.

(5) Let x_n, $n = 1, \cdots,$ and x be elements in $\mathcal{X}$. Then x_n converges to x weakly in $\mathcal{X}$ (i.e., $f(x_n)$ converges to $f(x)$ for any $f \in \mathcal{X}^*$) if and only if for any weakly open subset O containing x there exists an N such that $x_n \in O$ for any $n \geq N$.

4. Solve by Riese Representation Theorem the following boundary value problem for $f \in C[0, 1]$

$$u^{(4)} + u = f \text{ in } (0, 1),$$
$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

Provide all the necessary preparations. (You may assume all the results we proved in solving the Sturm-Liouville system.)

Solution of Problem 1. (1) To prove $\|\cdot\|_\infty$ is a norm, we shall only prove it satisfies the triangle inequality. There holds for any k

$$|x_k + y_k| \le |x_k| + |y_k| \le \|x\|_\infty + \|y\|_\infty.$$

By taking supremum over k, we get $\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$.

Now we prove $(l^\infty, \|\cdot\|_\infty)$ is complete. Suppose $\{x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots)\}$ is a Cauchy sequence in l^∞. By $|x_k^{(n)} - x_k^{(m)}| \le \|x^{(n)} - x^{(m)}\|_\infty$ for each fixed k, the scalar sequence $\{x_k^{(n)}\}$ is Cauchy and hence is convergent. For each k, we let $x_k^{(n)} \to x_k$ as $n \to \infty$. We shall prove $x = (x_1, x_2, \cdots) \in l^\infty$ and $\|x^{(n)} - x\|_\infty \to 0$ as $n \to \infty$. Note $\{x^{(n)}\}$ is bounded in l^∞ since it is Cauchy. Hence there exists an $M > 0$ such that $\|x^{(n)}\|_\infty \le M$ for any n. This implies in particular for any k

$$|x_k^{(n)}| \le M.$$

By letting $n \to \infty$, we get

$$|x_k| \le M.$$

Now we may take the supremum over k to conclude $\|x\|_\infty \le M$. The proof of the convergence $\|x^{(n)} - x\|_\infty \to 0$ is similar and is omitted.

(2) It is obvious that c_0 is a subspace. Now we prove c_0 is closed. To this end, we let $\{x^{(n)}\}$ be a sequence in c_0 which is convergent to x in l^∞. We need to prove $x_0 \in c$. Let $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdots)$ and $x = (x_1, x_2, \cdots)$. We need to prove $x_k \to 0$ as $k \to \infty$. To this end, we just note

$$|x_k| \le |x_k^{(n)} - x_k| + |x_k^{(n)}| \le \|x^{(n)} - x\|_\infty + |x_k^{(n)}|.$$

We first take a fixed n sufficiently large so that the first term is small. Then we take any k large so that the last term is small. This proves the convergence of the sequence $\{x_k\}$ to 0.

(3) For a fixed $x \in l^\infty$, we let $D = \limsup |x_k|$. For any $y = (y_1, y_2, \cdots) \in c_0$, $\lim_{k\to\infty} y_k = 0$, we have

$$\|x - y\|_\infty = \sup_k |x_k - y_k| \ge \limsup_{k\to\infty} |x_k - y_k| = \limsup_{k\to\infty} |x_k| = D.$$

Hence we have

$$dist(x, c_0) \ge D.$$

For each fixed n, set $y^{(n)} = (x_1, \cdots, x_n, 0, \cdots) \in c_0$. Then we have $x - y^{(n)} = (0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots)$ and

$$\|x - y^{(n)}\|_\infty = \sup_{k \ge n+1} |x_k| \to \limsup_k |x_k| = D, \quad \text{as } n \to \infty.$$

Hence $\{y^{(n)}\}$ is a minimizing sequence and $d(x, c_0) = D$.

Next, for each fixed $x \in l^\infty$ as above, consider $y = (y_1, y_2, \cdots)$ with

$$y_n = \begin{cases} x_n - D & \text{if } x_n > D \\ 0 & \text{if } |x_n| \le D \\ x_n + D & \text{if } x_n < -D. \end{cases}$$

Then it is easy to see that $y \in c_0$ and

$$x_n - y_n = \begin{cases} D & \text{if } x_n > D \\ x_n & \text{if } |x_n| \le D \\ -D & \text{if } x_n < -D. \end{cases}$$

Obviously $\|x - y\|_\infty = D$.

Therefore, $dist(x, c_0) = \limsup_{n\to\infty} |x_n|$ and this distance is realized by some element in c_0. □

Solution of Problem 2. By

$$|(Tx, x)| \le \|Tx\|\|x\|, \text{ and } |(Tx, x)| = |(x, T^*x)| \le \|T^*x\|\|x\|,$$

we get

$$c\|x\| \le \|Tx\|, c\|x\| \le \|T^*x\| \text{ for any } x \in \mathcal{H}.$$

This implies both T and T^* are injective, i.e., $\mathcal{N}(T) = \{0\}$ and $\mathcal{N}(T^*) = 0$. Next, recall $\overline{\mathcal{R}(T)} = \mathcal{N}(T^*)^\perp$. (This is a homework problem.) We obtain $\overline{\mathcal{R}(T)} = \mathcal{H}$. Now we prove $\mathcal{R}(T)$ is closed. To this end, we consider $y_n = Tx_n \in \mathcal{R}(T)$ such that $y_n \to y$ in $\mathcal{H}$. We shall prove $y \in \mathcal{R}(T)$. By $c\|x_n - x_m\| \le \|Tx_n - Tx_m\|$, we note $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$, and hence we may assume $x_n \to x$ in $\mathcal{H}$. By the continuity of T, we conclude $y_n = Tx_n \to Tx$, and hence $y = Tx \in \mathcal{R}(T)$. Therefore $\mathcal{R}(T) = \mathcal{H}$. Hence T is injective and surjective, and then has an inverse. By $c\|x\| \le \|Tx\|$ for any $x \in \mathcal{H}$, we get $c\|T^{-1}y\| \le \|y\|$ for any $y \in \mathcal{H}$. Therefore, $T^{-1} \in \mathcal{B}(\mathcal{H})$. (Note, we did not use the Banach Inverse Theorem.) □

Solution of Problem 3. We first note that $\mathcal{N}(f)$ is a closed subspace of codimension 1 for any nonzero $f \in \mathcal{X}^*$.

(1) This is straightforward and is omitted.

(2) Let O be weakly open and $x_0 \in O$ be an arbitrary point. Then there exist $f_1, \cdots, f_n \in \mathcal{X}^*$ such that

$$x_0 + \cap_{k=1}^n \{x \in \mathcal{X}; |f_k(x)| < 1\} \subset O.$$

Take $r = (\sup_{1\le k\le n}\{\|f_k\|\})^{-1}$. Then we have

$$B(0, r) \subset \cap_{k=1}^n \{x \in \mathcal{X}; |f_k(x)| < 1\}.$$

This implies $B(x_0, r) \subset O$. Hence O is open (in the norm topology). The other two statements follow easily.

(3) We shall prove the following first: Let $\mathcal{X}$ be a NVS and $f_1, \cdots, f_n, f \in \mathcal{X}^*$. Then $f \in \text{span}\{f_1, \cdots, f_n\}$ if and only if $\cap_{i=1}^n \mathcal{N}(f_i) \subset \mathcal{N}(f)$.

$\Rightarrow$ Suppose $f = \sum_{i=1}^n c_i f_i$. For any $x \in \cap_{i=1}^n \mathcal{N}(f_i)$, we have $f_i(x) = 0$, and then $f(x) = 0$, or $x \in \mathcal{N}(f)$.

$\Leftarrow$. We first consider whether

$$(\cap_{i \neq n} \mathcal{N}(f_i)) \cap \mathcal{N}(f_n)^c \neq \emptyset.$$

If $\cap_{i \neq n} \mathcal{N}(f_i) \subset \mathcal{N}(f_n)$, we simply drop f_n and consider $\{f_1, \cdots, f_{n-1}\}$. Otherwise, we consider whether

$$(\cap_{i \neq n-1} \mathcal{N}(f_i)) \cap \mathcal{N}(f_{n-1})^c \neq \emptyset.$$

We continue this process. Therefore we may assume for any $k = 1, \cdots, n$

$$(\cap_{i \neq k} \mathcal{N}(f_i)) \cap \mathcal{N}(f_k)^c \neq \emptyset.$$

Hence there exists an $x_k \in (\cap_{i \neq k} \mathcal{N}(f_i)) \cap \mathcal{N}(f_k)^c$, or

$$f_i(x_k) = 0 \text{ for } i \neq k, \quad f_k(x_k) = 1.$$

For any $x \in \mathcal{X}$, consider $y = x - \sum_{i=1}^n f_i(x) x_i$. Then $f_k(y) = f_k(x) - f_k(x) = 0$, or $y \in \cap_{i=1}^n \mathcal{N}(f_i)$. By the assumption, $y \in \mathcal{N}(f)$, or $f(y) = 0$. Hence we have $f(x) = \sum_{i=1}^n f_i(x) f(x_i)$, or $f = \sum_{i=1}^n f(x_i) f_i$.

Now we comeback to (3). We shall prove that any weakly open set containing 0 is unbounded. First, by the definition of weakly open set, there exist $f_1, \cdots, f_n \in \mathcal{X}^*$ such that

$$\{x \in \mathcal{X}; |f(x)| < 1\} \subset O.$$

In particular, we have

$$\cap_{i=1}^n \mathcal{N}(f_i) \subset O.$$

If $\cap_{i=1}^n \mathcal{N}(f_i) = \{0\}$, then $\cap_{i=1}^n \mathcal{N}(f_i) \subset \mathcal{N}(f)$ for any $f \in \mathcal{X}^*$. This implies $\mathcal{X}^*$ is spanned by $f_1, \cdots, f_n$, and hence finite dimensional. Then it is easy to see that $\mathcal{X}$ is finite dimensional, which leads to a contradiction. Therefore, we conclude that $\cap_{i=1}^n \mathcal{N}(f_i)$ is a nontrivial subspace, and hence unbounded. So O is unbounded.

(4) Let $\mathcal{M}$ be a closed subspace in $\mathcal{X}$. We shall prove that $\mathcal{M}^c$ is weakly open. For any $x_0 \notin \mathcal{M}$, there holds $\text{dist}(x_0, \mathcal{M}) > 0$. By Hahn-Banach Theorem, there exists an $f \in \mathcal{X}^*$ such that $f|\mathcal{M} = 0$ and $f(x_0) = 1$. This implies $x_0 + \{x \in \mathcal{X}; |f(x)| < 1\} \subset \mathcal{M}^c$, since $f(x_0 + x) = f(x_0) + f(x) = 1 + f(x) \neq 0$ if $|f(x)| < 1$. Hence $\mathcal{M}^c$ is weakly open.

(5) We shall only prove the case $x = 0$.

$\Rightarrow$ Let $\{x_m\}$ be a sequence convergent to 0 weakly. Consider any weakly open subset O of 0. Then there exist $f_1, \cdots, f_n \in \mathcal{X}^*$ such that

$$\cap_{k=1}^n \{x \in \mathcal{X}; |f_k(x)| < 1\} \subset O.$$

By the weak convergence of $\{x_m\}$ to 0, there holds $f_k(x_m) \to 0$ as $m \to \infty$ for any $k = 1, \cdots, n$. Then there exists an N such that $|f_k(x_m)| < 1$ for any $m \geq N$ and any $k = 1, \cdots, n$. This implies $x_m \in O$ for any $m \geq N$.

$\Leftarrow$ Let $\{x_m\}$ be a sequence such that for any weakly open set O containing 0 there exists an N such that $x_n \in O$ for any $m \geq N$. For any $f \in \mathcal{X}^*$, we shall prove $f(x_m) \to 0$. To this end, we consider an arbitrary $\varepsilon > 0$ and note that $O_\varepsilon = \{x \in \mathcal{X}; |f(x)| < \varepsilon\}$ is weakly open by (1). Hence there exists an N such that for any $m \geq N$ there holds $x_m \in O_\varepsilon$, or $|f(x_m)| < \varepsilon$. □

Solution of Problem 4. Set

$$X = \{v \in C^2[0,1]; v(0) = v(1) = v'(0) = v'(1) = 0\}.$$

For any $v \in X$, we multiply $\bar{v}$ to the equation and integrate by parts to get

$$\int_0^1 [u''\bar{v}'' + u\bar{v}]dx = \int_0^1 f\bar{v}dx.$$

For any $u, v \in X$, set

$$(u,v)_{\mathcal{H}^2} = \int_0^1 [u''\bar{v}'' + u\bar{v}]dx.$$

This is an inner product in X, and it induces a norm

$$\|u\|_{\mathcal{H}^2} = \left(\int_0^1 [|u''|^2 + |u|^2]dx\right)^{\frac{1}{2}}.$$

In the following, we also set

$$\|u\|_{\mathcal{L}^2} = \left(\int_0^1 |u|^2 dx\right)^{\frac{1}{2}},$$

$$\|u\|_{\mathcal{H}^1} = \left(\int_0^1 [|u'|^2 + |u|^2]dx\right)^{\frac{1}{2}},$$

and

$$\mathcal{H}_0^1(0,1) = \{u \in AC[0,1]; u' \in \mathcal{L}^2(0,1), u(0) = u(1) = 0\}.$$

We proved that $(\mathcal{H}_0^1(0,1), (\cdot,\cdot)_{\mathcal{H}^1})$ is a Hilbert space.

Step 1. We set

$$\begin{aligned}\mathcal{H}_0^2(0,1) &= \{u \in \mathcal{H}_0^1(0,1); u' \in \mathcal{H}_0^1(0,1)\} \\ &= \{u \in AC[0,1]; u' \in AC[0,1], u'' \in L^2(0,1), \\ &\qquad u(0) = u(1) = u'(0) = u'(1) = 0\}.\end{aligned}$$

We claim $(\mathcal{H}_0^2(0,1), (\cdot,\cdot)_{\mathcal{H}^2})$ is a Hilbert space. We only need to prove $\mathcal{H}_0^2(0,1)$ is complete with respect to the norm $\|\cdot\|_{\mathcal{H}^2}$.

Recall the estimate for any $u \in \mathcal{H}_0^1(0,1)$

$$\|u\|_{\mathcal{L}^2} \le \|u'\|_{\mathcal{L}^2}.$$

This implies for $u \in \mathcal{H}_0^2(0,1)$

$$\|u\|_{\mathcal{H}^1} + \|u'\|_{\mathcal{H}^1} \le 4\|u\|_{\mathcal{H}^2}.$$

Let $\{u_n\}$ be a Cauchy sequence in $\mathcal{H}_0^2(0,1)$. Then $\{u_n\}$ and $\{u_n'\}$ are Cauchy in $\mathcal{H}_0^1(0,1)$. There exist $u, v \in \mathcal{H}_0^1(0,1)$ such that $u_n \to u$ and $u_n' \to v$ in $\mathcal{H}_0^1(0,1)$. In particular, $u_n' \to u'$ and $u_n' \to v$ in $\mathcal{L}^2(0,1)$. This implies $u' = v$. Hence we have $u \in \mathcal{H}_0^2(0,1)$, and we can check easily $u_n \to u$ in $\mathcal{H}_0^2(0,1)$.

Remark. We may prove directly that the completion of X under the norm $\|\cdot\|_{\mathcal{H}^2}$ is exactly $\mathcal{H}_0^2(0,1)$.

Step 2. Define

$$F(v) = \int_0^1 v\bar{f}dx \text{ for any } v \in \mathcal{H}_0^2(0,1).$$

Then obviously F is a linear functional on $\mathcal{H}_0^2(0,1)$. Next, we have by Schwartz inequality

$$|F(v)| \le \|f\|_{\mathcal{L}^2}\|v\|_{\mathcal{L}^2} \le \|f\|_{\mathcal{L}^2}\|v\|_{\mathcal{H}_0^2(0,1)}.$$

Hence F is a bounded linear functional on $\mathcal{H}_0^2(0,1)$. By Riesz Representation Theorem, there exists a unique $u \in \mathcal{H}_0^2(0,1)$ such that $(u,\varphi)_{\mathcal{H}_0^2(0,1)} = F(\varphi)$ for any $\varphi \in \mathcal{H}_0^2(0,1)$, i.e.,

$$\int_0^1 [u''\bar{\varphi}'' + u\bar{\varphi}]dx = \int_0^1 f\bar{\varphi}dx \text{ for any } \varphi \in \mathcal{H}_0^2(0,1).$$

Step 3. By setting $v = f - u$, we obtain

$$\int_0^1 u''\bar{\varphi}''dx = \int_0^1 v\bar{\varphi}dx \text{ for any } \varphi \in \mathcal{H}_0^2(0,1). \tag{1}$$

We shall prove $u'' \in AC$, $u^{(3)} \in AC$, $u^{(4)} \in L^2$ and $u^{(4)} = v$. The crucial step is the following. We shall prove, under the assumption (1), there exists a $w \in \mathcal{L}^2(0,1)$ such that

$$\int_0^1 u''\bar{\psi}' dx = \int_0^1 w\psi dx \text{ for any } \psi \in \mathcal{H}_0^1(0,1). \tag{2}$$

Suppose this is already done. We conclude from (2) that $u'' \in AC(0,1)$ and $u^{(3)} = -w \in \mathcal{L}^2$. Now we may integrate by parts the left side of (1) to get

$$\int_0^1 u^{(3)}\bar{\varphi}' dx = -\int_0^1 v\bar{\varphi} dx \text{ for any } \varphi \in \mathcal{H}_0^2(0,1).$$

Then we get $u^{(3)} \in AC(0,1)$ and $u^{(4)} = v$, or $u^{(4)} + u = f$.

To prove (2), we set $g(x) = \int_{1/2}^x v(t)dt$. Then (1) implies

$$\int_0^1 u''\bar{\varphi}'' dx = -\int_0^1 g\bar{\varphi}' dx \text{ for any } \varphi \in \mathcal{H}_0^2(0,1). \tag{3}$$

Fix a $\phi \in \mathcal{H}_0^2(0,1)$ such that $\int_0^1 \phi = 1$. For any $\psi \in \mathcal{H}_0^1(0,1)$, consider

$$\varphi(x) = \int_0^x \psi - \int_0^1 \psi \int_0^x \phi.$$

It is easy to check that $\varphi \in \mathcal{H}_0^2(0,1)$. Substituting such a φ in (3), we get

$$\int_0^1 u''\bar{\psi}' = \int_0^1 \left(-g + \int_0^1 (u''\bar{\phi}') + \int_0^1 (g\bar{\phi}) \right) \bar{\psi}.$$

This finishes the proof of (2) with $w = -g + \int_0^1 (u''\bar{\phi}') + \int_0^1 (g\bar{\phi})$. $\square$