PROJECT: OPERATORS IN HILBERT SPACES

Prove all the following lemmas, theorems and examples.

1. Compact Operators

In the following, all spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are assumed to be Hilbert spaces.

Lemma 1.1. *Let $A : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Then the following statements are equivalent.*

(1) $A(B_1)$ is sequentially compact in $\mathcal{Y}$ where $B_1 = \{x \in \mathcal{X}; \|x\| \leq 1\}$.

(2) $A(B)$ is sequentially compact in $\mathcal{Y}$ where B is any bounded set in $\mathcal{X}$.

(3) For any bounded sequence $\{x_n\}$ in $\mathcal{X}$, $\{Ax_n\}$ has a convergent subsequence in $\mathcal{Y}$.

Definition 1.2. Such a linear operator is called compact. Let $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ = all compact linear operators from $\mathcal{X}$ to $\mathcal{Y}$.

Lemma 1.3. *There hold the following results.*

(1) $\mathcal{C}(\mathcal{X}, \mathcal{Y}) \subseteq \mathcal{B}(\mathcal{X}, \mathcal{Y})$.

(2) $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ is a linear subspace and closed in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

(3) If $A \in \mathcal{C}(\mathcal{X}, \mathcal{Y}), B \in \mathcal{B}(\mathcal{Y}, \mathcal{Z})$, then $BA \in \mathcal{C}(\mathcal{X}, \mathcal{Z})$. If $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y}), B \in \mathcal{C}(\mathcal{Y}, \mathcal{Z})$, then $BA \in \mathcal{C}(\mathcal{X}, \mathcal{Z})$.

(4) Let $A \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{M}$ be a closed subspace in $\mathcal{X}$. Then $A|\mathcal{M} \in \mathcal{C}(\mathcal{M}, \mathcal{Y})$.

(5) Let $A \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$. Then $\overline{\mathcal{R}(A)}$ is separable.

Theorem 1.4. *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then $A \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ if and only if $A^* \in \mathcal{C}(\mathcal{Y}, \mathcal{X})$.*

Hint: If $A \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$, then $AA^* \in \mathcal{C}(\mathcal{Y}, \mathcal{Y})$.

Now we provide several methods to verify whether a bounded linear operator is compact.

Definition 1.5. An operator $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is completely continuous if for any $x_n \rightharpoonup x$ weakly in $\mathcal{X}$ there holds $Ax_n \to Ax$ in norm in $\mathcal{Y}$.

Lemma 1.6. *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then $A \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ if and only if A is completely continuous.*

Definition 1.7. A linear operator $A : \mathcal{X} \to \mathcal{Y}$ has a finite rank if $\mathcal{R}(A)$ is finite dimensional. Let $\mathcal{C}_0(\mathcal{X}, \mathcal{Y})$ = all linear operators of finite rank.

Operators in $\mathcal{C}_0(\mathcal{X}, \mathcal{Y})$ have a simple formula.

Lemma 1.8. *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then $A \in \mathcal{C}_0(\mathcal{X}, \mathcal{Y})$ if and only if $A^* \in \mathcal{C}_0(\mathcal{Y}, \mathcal{X})$.*

Hint: Let $A \in \mathcal{C}_0(\mathcal{X}, \mathcal{Y})$. Then there exist $\{x_1, \cdots, x_n\} \subset \mathcal{X}$ and $\{y_1, \cdots, y_n\} \subset \mathcal{Y}$ such that

$$Ax = \sum_{i=1}^{n} (x, x_i) y_i \text{ for any } x \in \mathcal{X}.$$

Lemma 1.9. *(1) $\mathcal{C}_0(\mathcal{X},\mathcal{Y})$ is a linear space and $\mathcal{C}_0(\mathcal{X},\mathcal{Y}) \subseteq \mathcal{C}(\mathcal{X},\mathcal{Y})$.*
(2) $\mathcal{C}_0(\mathcal{X},\mathcal{Y})$ is dense in $\mathcal{C}(\mathcal{X},\mathcal{Y})$, i.e., $\overline{\mathcal{C}_0(\mathcal{X},\mathcal{Y})} = \mathcal{C}(\mathcal{X},\mathcal{Y})$.

Example 1.10. Let Ω be a bounded measurable subset in $\mathbb{R}^n$ and $K \in \mathcal{L}^2(\Omega\times\Omega)$. Define $A : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ by

$$Au(x) = \int_\Omega K(x,y)u(y)dy \text{ for any } u \in \mathcal{L}^2(\Omega).$$

Then A is a compact operator on $\mathcal{L}^2(\Omega)$.

2. Self-Ajoint Operators

In this section we always assume that $\mathcal{X}$ is a $\mathbb{C}$-Hilbert space.

Definition 2.1. $A \in \mathcal{B}(\mathcal{X})$ is self-adjoint if $A = A^*$, i.e., $(Ax, y) = (x, Ay)$ for any $x, y \in \mathcal{X}$.

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{X})$. Then A is self-adjoint if and only if $(Ax, x) \in \mathbb{R}$ for any $x \in \mathcal{X}$.*

Hint: Consider $(A(x + cy), x + cy)$ for $c = 1$ and $c = i$.

The preceding result is false if $\mathcal{X}$ is only assumed to be an $\mathbb{R}$-Hilbert space. Indeed for any operator A in an $\mathbb{R}$-Hilbert space, $(Ax, y) \in \mathbb{R}$.

Lemma 2.3. *Let $A \in \mathcal{B}(\mathcal{X})$ be self-adjoint. Then*

$$\|A\| = \sup\{|(Ax, x)|; \|x\| = 1\}.$$

Lemma 2.4. *Let $A \in \mathcal{B}(\mathcal{X})$ be self-adjoint and $\lambda \in \mathbb{C}$. Then $\lambda I - A$ is invertible in $\mathcal{B}(\mathcal{X})$ if and only if there exists a constant $c > 0$ such that*

$$\|(\lambda I - T)x\| \geq c\|x\| \text{ for any } x \in \mathcal{X}.$$

For a self-adjoint operator $A \in \mathcal{B}(\mathcal{X})$, we define

$$m_- = \inf_{x\in\mathcal{X},\|x\|=1} (Ax, x), \quad m_+ = \sup_{x\in\mathcal{X},\|x\|=1} (Ax, x).$$

Theorem 2.5. *Let $A \in \mathcal{B}(\mathcal{X})$ be self-adjoint.*
(1) For any $\lambda \notin \mathbb{R}$ the operator $\lambda I - A$ is invertible in $\mathcal{B}(\mathcal{X})$.
(2) For any $\lambda \in \mathbb{R} \setminus [m_-, m_+]$, $\lambda I - A$ is invertible in $\mathcal{B}(\mathcal{X})$.

Lemma 2.6. *Let $A \in \mathcal{B}(\mathcal{X})$ be self-adjoint such that $(Ax, x) \geq 0$ for any $x \in \mathcal{X}$. Then*
(1) $|(Ax, y)|^2 \leq (Ax, x)(Ay, y)$ for any $x, y \in \mathcal{X}$.
(2) $\|Ax\|^2 \leq \|A\|(Ax, x)$ for any $x \in \mathcal{X}$.

Corollary 2.7. *Let $A \in \mathcal{B}(\mathcal{X})$ be self-adjoint. Then $m_+I - A$ and $m_-I - A$ are not invertible in $\mathcal{B}(\mathcal{X})$.*

Theorem 2.8. *Let $A \in \mathcal{B}(\mathcal{X})$ be self-adjoint.*
(1) If there exists an $x_- \in \mathcal{X}$ with $\|x_-\| = 1$ and $(Ax_-, x_-) = m_-$, then m_- is an eigenvalue of A and x_- is a corresponding eigenvector.
(1) If there exists an $x_+ \in \mathcal{X}$ with $\|x_+\| = 1$ and $(Ax_+, x_+) = m_+$, then m_+ is an eigenvalue of A and x_+ is a corresponding eigenvector.