Complex Analysis, Examination 1, Math 505

October14, 2002

1. Let (S, d) be a metric space. (a) Define, in terms of d, what is meant by saying that a subset U of S is an open set. (b) Define what is meant by saying that $f : S \longrightarrow \mathbb{R}$ is a continuous mapping. (c) Fix a point $p \in S$ and define $g : S \longrightarrow \mathbb{R}$ by

$$g(x) = \sin\{(d(x,p))^2\}.$$

Show that g is continuous on S. (You may assume that the sine function is continuous on $\mathbb{R}$)

2. Let (S, d) be a metric space and K a subset of S. (a) Define what is meant by saying K is compact (b) Show, from your definition of compactness, that a closed subset of a compact set is compact. (c) Show that the intersection of a nested decreasing sequence of compact sets is a nonempty compact set. Must it be connected?

3. Let $\gamma : [\alpha, \beta] \longrightarrow \Omega$ be a piecewise smooth curve in a region Ω on which p and q are continuous real or complex valued functions. (a) Write down precisely, as a definite integral, what is meant by $\int_\gamma p dx + q dy$.

(b) If $F(x, y) = \int_\gamma p dx + q dy$ is independent of the piecewise smooth path γ chosen within Ω from some fixed point $(x_0, y_0) \in \Omega$ to any $(x, y) \in \Omega$, then compute $\frac{\partial F}{\partial z}$. (c) If $f(z)$ is an analytic function on $\mathbb{C}$ with real part $u = x^2 - y^2$, then determine an expression for $f(z)$. Give reasons and write your answer in terms of the variable z.

4. Let f be an analytic function on the open unit disk D centred at the origin. Explaining any notation introduced, (a) state Cauchy's Theorem for f, (b) state the Cauchy integral formula for f, (c) evaluate $\int_\gamma \frac{f(z)}{z-1} dz$ where γ is any piecewise smooth closed curve in D.