Assignment 10
Math 605, Fall '00

Read sections 7.1–7.4 from Greene and Krantz.

Solve the following problems.

1. From the textbook. Pages 203–207: 3 (example only), 10, 17

2. Let $\mathbf{H} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$ be the upper half-plane. Prove that $T \in \operatorname{Aut}(\mathbf{H})$ if and only if $T(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbf{R}$ and $ad - bc > 0$. *Hint: the fact that* $\operatorname{Aut}(D(0,1))$ *consists only of linear fractional transformations can be used to show the same about* $\operatorname{Aut} H$.

3. Let $d_P(z, w)$ denote the Poincaré distance between points $z, w \in D(0,1)$ and $D_P(z,r) = \{w \in D(0,1) : d_P(z,w) < r\}$. Explain why $D_P(z,r)$ is really just $D(z', r')$ for some $z' \in D(0,1)$ and $r' > 0$ and why if r is fixed and $|z| \to 1$, we must have $r' \to 0$. *Hint: to avoid nightmarish computation, rely on the fact that* $T \in \operatorname{Aut}(D(0,1)$ *implies that* T *preserves the Poincaré distance between points.*

4. Dynamics of Holomorphic Maps of the Disk. Let $f : D(0,1) \circlearrowleft$ be holomorphic but not an automorphism, and $f^n = f \circ f \circ f \circ \cdots \circ f$, n times. Use the fact that f decreases Poincaré distances to prove the following.

- Given $z_0 \in D(0,1)$, we have $\lim_{n\to\infty} |f^n(z_0)| = 1$ if and only if the same is true for any $z \in D(0,1)$.
- Any "periodic point" $z_0 \in D(0,1)$ of f is actually "fixed." That is, $f^n(z_0) = z_0$ for some $n > 0$ implies $f(z_0) = z_0$.
- If $z_0 \in D(0,1)$ is fixed by f, then $\lim_{n\to\infty} f^n(z) = z_0$ uniformly on compact subsets of $D(0,1)$.

5. Suppose that $U \subset \mathbf{C}$ is open and $\mathcal{F}$ is a family of holomorphic maps $f : U \to \mathbf{C}$. Let $W = \bigcup_{f\in\mathcal{F}} f(U)$, and suppose that W is not dense in $\mathbf{C}$. Show that $\mathcal{F}$ is *normal* in the following generalized sense: for any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that either

- f_{n_j} converges uniformly on compacts subsets of U to a holomorphic function $f : U \to \mathbf{C}$;
- or $\lim_{j\to\infty} |f_{n_j}(z)| = \infty$ uniformly on compact subsets of U.