Assignment 12 (Due the Friday after Thanksgiving)
Math 605, Fall '00

Read sections 7.8–7.9 from Greene and Krantz.

Solve the following problems.

1. From the textbook. Pages 244–254: 19 (try to avoid using holomorphic functions), 47 (to make life easier, assume that F is *conformal*), 73 (note that parts (d) and (e) are closely related to problem 3 below)

2. (A Crash Course in Conjugate Differentials) Recall the following facts:

- (From this class) If $U \subset \mathbf{C}$ is open and $f : U \to \mathbf{C}$ is holomorphic, then f has a holomorphic antiderivative F on U if and only if $\int_\gamma f(z)\,dz = 0$ around every piecewise smooth closed curve $\gamma : [a,b] \to U$. Given $F(z_0)$ for some $z_0 \in U$, we obtain $F(z_1)$ by choosing a path from z_0 to z_1 and setting
$$F(z_1) = F(z_0) + \int_\gamma f(z)\,dz.$$
- (From vector calculus) If $U \subset \mathbf{R}^2$ is open, $h : U \to \mathbf{R}^2$ is C^1, and $\gamma : [a,b] \to U$ is a piecewise smooth curve, then
$$h(\gamma(b)) - h(\gamma(a)) = \int_\gamma h_x\,dx + h_y\,dy.$$
The "1-form" $h_x\,dx + h_y\,dy$ is collectively abbreviated as dh (the differential of h). The conjugate of a 1-form $\omega = A\,dx + B\,dy$ is the 1-form $*\omega \stackrel{\text{def}}{=} -B\,dx + A\,dy$.

Use these to prove the following for $h : U \to \mathbf{R}$ harmonic.

- h has a harmonic conjugate $h^* : U \to \mathbf{C}$ if and only if $\int_\gamma *dh = 0$ for every piecewise smooth closed curve $\gamma : [a,b] \to U$.
- If $\gamma_1, \gamma_2 : [a,b] \to U$ are homologous closed curves, then
$$\int_{\gamma_1} *dh = \int_{\gamma_2} *dh.$$
- If $h : A \to \mathbf{C}$ is harmonic, where $A = \{z \in \mathbf{C} : R_1 < |z| < R_2\}$ is an annulus, then for every r between R_1 and R_2
$$\int_0^{2\pi} h(re^{i\theta})\,d\theta = \alpha \log r + \beta,$$
where $\alpha, \beta \in \mathbf{R}$ are constants and
$$\alpha = \int_{|z|=r} *dh.$$
(*Hint: consider the difference* $h(z) - \alpha \log |z|$ *for appropriately chosen* α.)
- Suppose that $h : D^*(0,1) \to \mathbf{R}$ is harmonic and bounded. Show that h extends to a harmonic function on $D(0,1)$. What happens if we assume only that h is bounded above?