Assignment 2
Math 605, Fall ‘00

Read sections 2.1, 2.3-2.4, and 3.2 from Greene and Krantz.

Solve the following problems.

1. Let $U \subset \mathbf{C}$ be connected and open and $u : U \to \mathbf{R}$, $f : U \to \mathbf{C}$ be an harmonic and an analytic function, respectively. Show that the harmonic conjugate for u and the anti-derivative for f are unique (if they exist) up to additive constants on U.

2. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be linear.

- Show that the following two conditions are equivalent
  - there exists a constant $r > 0$ such that $||T(v)|| = r||v||$ for every vector $v \in \mathbf{R}^n$.
  - The angle $\angle(v, w)$ is the same as $\angle(T(v), T(w))$ for every $v, w \in \mathbf{R}^n$.
- T is called *conformal* if either (and hence both) of the above two conditions hold and T preserves orientation—i.e. has matrix with positive determinant. Show that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is conformal if and only if it's complex linear and invertible.
- Show that if $T : \mathbf{R}^2 \to \mathbf{R}^2$ is conformal, it can be factored into a ‘scaling’ times a ‘rotation’—i.e. the matrix for T factors as
$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
  What is the relationship between r, θ and A?

3. Complete the following steps in a discussion of the antiderivative for $1/z$.

- Verify that $u(z) = \log|z|$ is harmonic when $z \neq 0$ and that $v(z) = \tan^{-1}(y/x)$ is a harmonic conjugate for $\log|z|$ when $z = x + iy$ with $x \neq 0$.
- Show that the domain of v can be extended to give a harmonic conjugate for $\log|z|$ on $\mathbf{C} \setminus (-\infty, 0]$.
- Show that $\log z \stackrel{\text{def}}{=} u+iv$ is an antiderivative for $1/z$. This function is called the *principle branch* of the logarithm function.
- Show that $1/z$ cannot have an antiderivative defined on all of $\mathbf{C} \setminus \{0\}$.
- Show that with appropriate restrictions on domain, e^z and $\log z$ are inverse functions of one another.

4. Let $U, V \subset \mathbf{C}$ be open sets, $f : U \to V$ complex analytic, and $u : V \to \mathbf{R}$ harmonic. Show that $u \circ f$ is harmonic. (Hint: an enlightened proof avoids direct use of the chain rule.)

5. Suppose that $U \subset \mathbf{C}$ is open and $f : U \to \mathbf{C}$ is complex differentiable (not necessarily C^1) at $z_0 \in U$. Show by letting $h \to 0$ from different directions in the limit defining $f'(z_0)$ that f satisfies the Cauchy-Riemann equations at z_0. That is, if $f = u + iv$, show $u_x = v_y$ and $u_y = -v_x$ at z_0.