Assignment 4
Math 605, Fall ‘00

Read sections 3.4, 3.6, 4.1, 4.2 from Greene and Krantz.

Solve the following problems.

1. From the textbook. Pages 96-105: 20bc, 22

2. Suppose that $U \subset \mathbf{C}$ is open and connected, $f_n : U \to \mathbf{C}$, $n \in \mathbf{N}$, are holomorphic functions converging uniformly on compact subsets of U to a function f, and that for each $n \in \mathbf{N}$ $F_n : U \to \mathbf{C}$ is an anti-derivative of f_n. Show that the functions F_n converge uniformly to an anti-derivative of f, provided that for some $z_0 \in U$ the sequence $\{F_n(z_0)\}_{n=0}^{\infty}$ converges.

3. Consider the power series $\sum_{n=0}^{\infty} a_n z^n$ where $a_0 = 1, a_1 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$ (i.e. $\{a_n\}$ is the Fibonacci sequence).

- Show that the radius of convergence for this series is at least 1/2.
- Show that on the open disk where the series converges, it converges to a rational function. (Hint: use a trick similar to the one we used to compute $\sum_{n=1}^{\infty} z^n$—i.e. compare the partial sum S_n with zS_n, etc. in order to obtain a formula for S_n.)
- Use your answer to the second part to compute the actual radius of convergence of the series? Does this fit with anything else you know about Fibonacci numbers?

4. The function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is C^∞ (i.e. infinitely differentiable) but not equal to its Taylor series centered at 0. Many textbooks point this out with a dismissive “one can easily show that...” thrown in as an excuse for not proving their asserion. Now it’s pretty clear that f is infinitely differentiable for $x \neq 0$ and it’s plausible that all derivatives of f exist and equal zero when $x = 0$, but I have yet to see an “easy” demonstration of the latter fact. So either show me an easy proof, or complete the following outline to obtain mine:

- Show by induction that $\lim_{x\to 0} x^n e^{-1/x^2} = 0$ for all $n \in \mathbf{Z}$. L’Hôpital’s rule from Calculus helps here, but only after a change of variables. Work only with a righthand limit if it makes your life easier.
- Show by induction that $\frac{d^n}{dx^n} e^{-1/x^2} = R_n(x)$ for some rational function $R_n(x)$ and all $n \in \mathbf{N}$.
- Using these facts, show that all derivatives of f exist and are zero at $x = 0$.
- Now explain why there is no interval containing 0 on which $f(x)$ is equal to some convergent power series $\sum_{n=0}^{\infty} c_n x^n$.