Assignment 6
Math 605, Fall '00

Read sections 4.2–4.7 from Greene and Krantz.

Solve the following problems.

1. From the textbook. Pages 147–158: 33.

2. *Conformality of the stereographic projection.*

- Show that the stereographic projection $\pi : S^2 \to \overline{\mathbf{C}}$ is conformal. That is, show that if $p \in S^2$ (don't worry about the case where $p = (0,0,1)$), then $||D\pi_p(v)|| = r(p)||v||$ for every vector $v \in \mathbf{R}^3$ tangent to S^2 at p and some $r(p)$ independent of v.
- Give a formula for $r(p)$ in terms of the image $z = \pi(p)$ instead of p. Given a C^1 curve $\tilde{\gamma} : [a,b] \to S^2$ and its image $\gamma = \pi \circ \gamma : [a,b] \to \mathbf{C}$, how would you express the length of $\tilde{\gamma}$ as an integral over γ?

3. Complex analysts tend to lump lines and circles in $\mathbf{C}$ into the same category. Think like a complex analyst and show that the stereographic projection maps a circle S on S^2 to

- a circle in $\mathbf{C}$ if $(0,0,1) \notin S$; and
- a line in $\mathbf{C}$ if $(0,0,1) \in S$.

It might help to remember that every circle on S^2 is obtained by intersecting S^2 with a plane.

4. *Homogeneous coordinates for the Riemann Sphere.* Consider the quotient space $\mathbf{P}^1 \stackrel{\text{def}}{=} (\mathbf{C}^2 \setminus \{0\})/\sim$, where $(z_1, w_1) \sim (z_2, w_2)$ if and only if $z_1 w_2 = z_2 w_1$—i.e. if either pair is a (possibly complex) multiple of the other.

- Show that the map $\pi : (\mathbf{C}^2 \setminus \{0\}) \to \overline{\mathbf{C}}$ given by $\pi(z,w) = z/w$ ($= \infty$ if $w = 0$) induces to a well-defined homeomorphism $\tilde{\pi} : \mathbf{P}^1 \to \overline{\mathbf{C}}$.
- A *homogeneous map* of $\mathbf{C}^2$ is a map $F : \mathbf{C}^2 \to \mathbf{C}^2$ such that
$$F(z,w) = (F_1(z,w), F_2(z,w))$$
where F_1 and F_2 are homogeneous polynomials of the same degree. It turns out that for any rational function $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$, one can find a homogeneous map $F : \mathbf{C}^2 \to \mathbf{C}^2$ such that $f \circ \pi = \pi \circ F$. Illustrate this fact using a reasonably generic example f of a rational map.
- Show (again by example) that one can likewise associate to each homogeneous map $F : \mathbf{C}^2 \to \mathbf{C}^2$ a rational map $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$