Assignment 7
Math 605, Fall '00

Read sections 4.7–5.5 from Greene and Krantz.

Solve the following problems.

1. From the textbook. Pages 147–158: 29bh,

2. The Laurent series for a holomorphic function depends on more than where you center your annulus. Consider, for instance, the function $f(z) = 1/\sin z$.

- Compute the principle part of the Laurent series for $f(z)$ on $D^*(0, \pi)$.
- Find a rational function $R(z)$ such that $1/\sin z - R(z)$ has removable singularities on on $|z| < 2\pi$.
- Compute the coefficients of the negative powers of z in the Laurent series expansion for $1/\sin z$ on the annulus $\{\pi < |z| < 2\pi\}$.

3. Suppose that $U \subset \mathbf{C}$ is homologically trivial and that $f : U \to \mathbf{C}$ is holomorphic and non-vanishing at every point in U. Show that for every $k \in \mathbf{N}$, there exists a holomorphic function $g : U \to \mathbf{C}$ such that $f = g^k$.

Not for credit (*but I'd be happy to check your solution anyhow*): The annulus $A = \{1 < |z| < 2\}$ is not homologically trivial. Nevertheless, some non-vanishing holomorphic functions $f : A \to \mathbf{C}$ admit square roots—i.e. holomorphic functions $g : A \to \mathbf{C}$ such that $f = g^2$. $f(z) = z^{-2}$ is an obvious example.

- Give a necessary and sufficient condition for a non-vanishing holomorphic function $f : A \to \mathbf{C}$ to have a kth root.
- Use your condition to show that $f(z) = \frac{z+10}{z^2+1}$ admits a square root on A.