Assignment 9
Math 605, Fall '00

Read sections 6.1–6.3 from Greene and Krantz.

Solve the following problems.

1. Recall the quotient map $\pi : \mathbf{C}^2 \setminus \{\mathbf{0}\} \to \overline{\mathbf{C}}$ introduced in Homework 6, problem 4. Let $T \in \mathrm{Aut}(\mathbf{C})$ be a linear fractional transformation and $\tilde{T} : \mathbf{C}^2 \to \mathbf{C}^2$ the associated linear transformation.

- Show that $p \in \mathbf{C}^2 \setminus \{0\}$ is an eigenvector of $\tilde{T}$, if and only if $z = \pi(p)$ is a fixed point of T—i.e. $T(z) = z$.
- Show that any linear fractional transformation T can be conjugated by another linear fractional transformation S so that $S \circ T \circ S^{-1}(z)$ is either az for some $a \in \mathbf{C} \setminus \{0\}$ or $z + 1$.
- Suppose that $T(z) = \frac{z-2}{3z-4}$. Give an exact formula for $T^n(z)$ (meaning T composed with itself n times and applied to z).

2. Show that $f \in \mathrm{Aut}(D^*(0,1))$ if and only if $f(z) = e^{i\theta} z$ for some $\theta \in \mathbf{R}$.

3. Suppose that $U, V \subset \mathbf{C}$ are domains and that $f : U \to V$ is a holomorphic bijection. Then there exists an inverse function $f^{-1} : V \to U$. We never actually proved that f^{-1} is holomorphic, so we'll fix that now. Show that:

- given $w_0 \in U$, there exist numbers $\epsilon, \delta > 0$ such that $w \in D(w_0, \epsilon)$ implies
$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|z-z_0|=\delta} \frac{z f'(z)}{f(z) - w}\, dz,$$
where $f(z_0) = w_0$;
- (using this formula), f^{-1} is holomorphic on $D(w_0, \epsilon)$; and since w_0 was arbitrary, f^{-1} is holomorphic on V.

(This is essentially problem 11 on page 178 of Greene and Krantz.)

4. Find a (sequence of) conformal map(s) taking the set A onto the set B when

- $A = \mathbf{C} \setminus \{e^{i\theta} : \theta \in [0, \pi]\}$, $B = \mathbf{C} \setminus D(0,1)$;
- $A = \mathbf{C} \setminus ((-\infty, 0] \cup \{1\})$, $B = \mathbf{C} \setminus ((-\infty, 0] \cup \{i\})$.
- $A = \{x + iy \in \mathbf{C} : -1 < x - y < 1\}$, $B = \mathbf{H}$.

You should give a formula for each map in the sequence, but you can just draw pictures to illustrate the effect of the map.