Final Exam (Due Tuesday, December 12 by 10:30 AM)
Math 605, Fall '00

Solve the following problems. You can use your notes, the textbook, old homework solutions, and your professor as resources, but nothing else. Do not, in particular, talk to other people about the exam or consult with other books.

1. Find a sequence of conformal maps sending $D(0,1) \cap D(1,1)$ biholomorphically onto $\mathbf{H}$. You needn't justify the pictures that you draw, but make sure they're correct (i.e. make sure that the maps you give do what you want them to do).

2. Let $f : U \to \mathbf{C}$ be a non-constant holomorphic function on a domain $U \subset \mathbf{C}$. Let $N(z) = z - f(z)/f'(z)$. Show that if $f(z_0) = 0$, then there is a disk $D(z_0, r)$ to which $N(z)$ extends as a well-defined holomorphic function satisfying $N(z_0) = z_0$. Then show for $r > 0$ small enough, that

$$\lim_{n\to\infty} N^n(z) \to z_0$$

uniformly for $z \in D(z_0, r)$. Note that $f'(z_0)$ *can* vanish and be careful to explain why the converge is uniform rather than merely pointwise.

3. Let $h : U \to \mathbf{R}$ be a harmonic function on a domain $U \subset \mathbf{C}$. Show that if h vanishes on some open subset of U, then $h \equiv 0$ on U.

4. Explain why there exists a holomorphic function $f : \mathbf{C} \setminus [-1, 1] \to \mathbf{C}$ such that $e^{f(z)} = \frac{z-1}{z+1}$.

5. Let $R > r > 0$ be fixed numbers and $A = \{z \in \mathbf{C} : r < |z| < R\}$ be an annulus. A function $f : A \to \mathbf{R}$ is said to be *radially symmetric* if $f(z) = \varphi(|z|)$ for some function $\varphi : (r, R) \to \mathbf{R}$. Show that every radially symmetric harmonic function $h : A \to \mathbf{R}$ has the form $h(z) = \alpha \log |z| + \beta$ for some $\alpha, \beta \in \mathbf{R}$.

6. Consider the domains

$$\Omega_n = D(0,1) \cup D(3,1) \cup \{x + iy \in \mathbf{C} : x \in [0,3], y \in (-1/n, 1/n)\}.$$

Let $f_n : D(0,1) \to \Omega_n$ be a sequence of biholomorphic maps such that $f_n(0) = 0$ for every $n \in \mathbf{N}$. Show that there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges uniformly on compact subsets of $D(0,1)$ to an automorphism of $D(0,1)$.