Math 605 Midterm Exam
October 11, 2000

1. State the following (5 points each):

- The definition of a complex analytic function $f : U \to \mathbf{C}$ where U is open.
- Cauchy's Integral formula (either the first version or the general version we proved later).
- The definition of e^z for $z \in \mathbf{C}$.
- Liouville's Theorem.

2. Suppose that $\sum_{n=0}^{\infty} a_n(z+2)^n$ is a power series for $\frac{z^2-1}{z^2+(1+i)z}$. What is $\limsup_{n\to\infty} |a_n|^{1/n}$? (10 points)

3. At what points z is the function $f(z) = |z|^2$ complex differentiable? (10 points)

4. Let γ be a C^1 parametrization of a triangle (in the counterclockwise direction) with one vertex at the origin, one vertex at 1 and the third vertex at $w = a + bi$ where $b > 0$.

- Show by direct computation that $\int_\gamma \bar{z}\, dz$ gives 2i times the area of the triangle. (10 points)
- Suppose we pick a complex number $A = re^{i\theta}$ and change the triangle by multiplying all its vertices by A. How does the integral in the first part change?

5. Suppose that $f : \mathbf{C} \to \mathbf{C}$ is an entire function. Show that f is also an *isometry*—i.e.

$$|f(z) - f(w)| = |z - w|$$

for every $z, w \in \mathbf{C}$—if and only if $f(z) = Az + B$ where $A, B \in \mathbf{C}$ are constants and $|A| = 1$. (20 points)

6. Suppose that $f : D^*(0,1) \to \mathbf{C}$ is holomorphic and satisfies

$$z^2 f'(z) = [f(z)]^2 + z.$$

Show that f has an essential singularity at 0. (20 points)