Assignment 1

(a) If $x \in \sigma$ and $x \neq a_0, \ldots, a_n$, then x lies in some open line segment contained in σ. (Assume $n > 0$.)
Remark: You do not have to assume $n > 0$ since if $n = 0$ there are no points $x \in \sigma$ with $x \neq a_0$ if $n = 0$.
Write $x = \Sigma_{i=0}^n t_i a_i$ with $t_i \geq 0$ and $\Sigma_{i=0}^n t_i = 1$. Since no $t_i = 1$, there must be a j with $0 < t_j < 1$ (indeed there must be at least two such). Then

$$t a_j + \sum_{i=0, i\neq j}^{n} \frac{(1-t)}{1-t_j} t_i a_i \qquad 0 \leq t \leq 1$$

is a line segment in σ and x lies on this segment at time $t = t_j$.

(b) Show that a_0 lies in no open line segment contained in σ by showing that if $a_0 = tx + (1-t)y$, where $x, y \in \sigma$ and $0 < t < 1$, then $x = y = a_0$.
Let $x = \Sigma_{i=0}^n t_i a_i$ and $y = \Sigma_{i=0}^n s_i a_i$ be the coordinates of x and y. Then $a_0 = \Sigma_{i=0}^n \big(t t_i + (1-t)s_i\big) a_i$. Because the coordinate representation is unique, $t t_0 + (1-t)s_0 = 1$ and $t t_i + (1-t)s_i = 0$ for $i > 0$. Since $t_i \geq 0$, $s_i \geq 0$ and $0 < t < 1$, the equation $t t_i + (1-t)s_i = 0$ implies $t_i = s_i = 0$. Hence $t_0 = s_0 = 1$.

Remark: This exercise is the basis for the simplex method in linear programing. First note that a linear function on a line segment is either constant or takes on its maximum value at one end and its minimum value at the other. Hence this exercise shows that a linear function on $\mathbf{R}^n$ restricted to a simplicial complex achieves its maximum and its minimum at vertices.