Prob. 3, p. 213. Let X be a space and let X_C denote X with the topology coherent with the collection of compact subsets..

(a) If $B \subset X$, show B is compact as a subspace of X if and only if it is compact as a subspace fo X_C.

Proof: Let B denote the set B with the subspace topology from X and use B_C to denote B with the subspace topology from X_C. The function $i\colon X_C \to X$ which is the identity as a map of sets is continuous. To see this, let $A \subset X$ be closed. We need to see $i^{-1}(A) \subset X_C$ is closed, which is equivalent to checking that $A \cap C$ is a closed subset of C for every compact subset $C \subset X$. But $A \cap C$ is displayed as C intersected with a closed subset of X so it is closed in the subspace topology on C. Now i restricts to a continuous bijection $B_C \to B$. If B_C is compact, so is B since it is the image of a compact set. Now suppose B is compact. If $A \subset B_C$ is closed, this means there exits $\bar{A} \subset X_C$ which is closed so that $A = \bar{A} \cap B_C$. But $\bar{A}$ is closed in X_C so $\bar{A} \cap D$ is closed in D for every compact subset $D \subset X$. Taking $D = B$, we see $\bar{A} \cap B$ is closed in B. But $\bar{A} \cap B = A$, so A is closed in B. Hence $i\colon B_C \to B$ is a closed map, hence a homeomorphism. Since B is compact, so is B_C.

A space is called *compactly generated* provided the natural map $i_X\colon X_C \to X$ is a homeomorphism.

(b) X_C is compactly generated.

Proof: We show the continuous bijection $i_{X_C}\colon (X_C)_C \to X_C$ is closed which shows that it is a homeomorphism. Let $A \subset (X_C)_C$ be closed. This means $A \cap D$ is closed in D for every compact subset $D \subset X_C$. By part(a), every compact subset $E \subset X$ is compact in X_C so $A \cap E$ is closed in E. By definition, A is closed in X_C.

Remark: Let $f\colon Y \to X$ is a map. Let $f_C\colon Y_C \to X_C$ be the map induced on the sets by f. Then f_C is continuous. To show this, let $A \subset X_C$ be closed and consider $f_C^{-1}(A) = f^{-1}(A)$. Let $D \subset Y$ be compact: then so is $f(D)$. Observe $f^{-1}(A) \cap D = f^{-1}\big(A \cap f(D)\big) \cap D$. Now $A \cap f(D)$ is closed in $f(D)$ so $f^{-1}\big(A \cap f(D)\big)$ is closed in $f^{-1}\big(f(D)\big)$. It follows that $f^{-1}\big(A \cap f(D)\big) \cap D$ is closed in $f^{-1}\big(f(D)\big) \cap D$, hence $f^{-1}(A) \cap D$ is closed in D. But this means $f^{-1}(A)$ is closed in Y_C and hence f_C is continuous.

(c) Show that the inclusion $i_X\colon X_C \to X$ induces an isomorphism in singular homology.

Proof: Consider the chain map $(i_X)_\#\colon S_*(X_C) \to S_*(X)$. Since i_X is one to one, so is $(i_X)_\#$. To see that $(i_X)_\#$ is onto, let $T\colon \Delta^r \to X$ be any singular simplex. By part (a), $\Delta^r_C = \Delta^r$ and the composition $\Delta^r \xrightarrow{i^{-1}_{\Delta^r}} \Delta^r_C \xrightarrow{T_C} X_C \xrightarrow{i_X} X$ is just T. Hence $(i_C)_\#(T_C) = T$ so $(i_C)_\#$ is onto. Since the chain complexes are isomorphic via $(i_C)_\#$, $(i_X)_*$ induces an isomorphism in singular homology.

(d) If $X \times_C Y = (X \times Y)_C$, and if K_1 and K_2 are simplicial complexes, show that $|K| \times_C |L|$ is coherent with the subspaces $\sigma \times \tau$.

Proof: By Lemma 2.5 p.10 of the book, any compact subset of $|K|$ or $|L|$ is contained in a finite subcomplex. If $D \subset |K| \times |L|$ and if $\pi_K\colon |K| \times |L| \to |K|$ and $\pi_L\colon |K| \times |L| \to |L|$ denote the projections, $D \subset \pi_K(D) \times \pi_L(D)$. Hence if D is compact so are $\pi_K(D)$ and

$\pi_L(D)$ and hence there are finite subcomplexes K_0 and L_0 so that $D \subset |K_0| \times |L_0|$. Moreover $D \subset \bigcup_{\sigma \in K_0, \tau \in L_0} \sigma \times \tau$. We will use this in a moment.

First, let $A \subset |K| \times_C |L|$ be closed. This means that for every compact $D \subset |K| \times |L|$, $A \cap D$ is a closed subset of D. But $\sigma \times \tau$ is a product of compact sets, hence compact so consider the case $D = \sigma \times \tau$. Hence $A \cap (\sigma \times \tau)$ is a closed subset of $\sigma \times \tau$. By definition, A is closed in the topology on $|K| \times |L|$ coherent for $\sigma \times \tau$.

Conversely, suppose A is closed in the topology on $|K| \times |L|$ coherent for the $\sigma \times \tau$: i.e. $A \cap (\sigma \times \tau)$ is closed in $\sigma \times \tau$. If D is compact, $D \subset \bigcup_{\sigma \in K_0, \tau \in L_0} \sigma \times \tau$, so $A \cap D = D \cap \big(\bigcup_{\sigma \in K_0, \tau \in L_0} (\sigma \times \tau) \cap A \big)$. Each $(\sigma \times \tau) \cap A$ is closed in $\sigma \times \tau$ and there are only finitely many elements in the union, so $\cup_{\sigma \in K_0, \tau \in L_0} (\sigma \times \tau) \cap A$ is closed in $\cup_{\sigma \in K_0, \tau \in L_0} \sigma \times \tau$. But $\cup_{\sigma \in K_0, \tau \in L_0} \sigma \times \tau$ is a compact subset of a Hausdorff space, hence closed in $|K| \times |L|$. Hence $\cup_{\sigma \in K_0, \tau \in L_0} (\sigma \times \tau) \cap A$ is closed in $|K| \times |L|$. It follows that $A \cap D$ is closed in D and therefore A is closed in $|K| \times_C |L|$.