Prob. 1 & 3, p. 221.

1. Let X be a CW complex and $A \subset X$ a compact subset.
   (a) Show that A intersects only finitely many open cells of X. Where do you use "closure–finiteness"?

Proof: To help with this problem as well as part (b) to come, put a partial ordering on the open cells of X as follows: $e_\alpha \prec e_\beta$ provided $\bar{e}_\beta \cap e_\alpha \neq \emptyset$ and $\dim e_\alpha < \dim e_\beta$. Write $e_\alpha < e_\beta$ provided there exits a sequence of cells $e_{\alpha_0} = e_\alpha$, e_{α_1}, $\ldots e_{\alpha_r} = e_\beta$ such that $e_{\alpha_i} \prec e_{\alpha_{i+1}}$ for $0 \leq i < r$. Since the dimension of a cell is finite, any chain $\cdots \prec e_{\alpha_r} \prec \cdots$ must have finite length. Indeed, the maximal such length is the dimension plus 1. Since $e_\alpha \prec e_\beta$ if and only if $\dot{e}_\beta \cap e_\alpha \neq \emptyset$. If we fix e_β, "closure–finiteness" implies that there are only finitely many open cells e_α with $e_\alpha \prec e_\beta$. It follows that the set of open cells e_α with $e_\alpha < e_\beta$ is also finite. This is a general result about partial orderings. Let $\mathcal{I}(e_\beta, 0) = \{e_\beta\}$. Define $\mathcal{I}(e_\beta, r)$ inductively by $\mathcal{I}(e_\beta, r) = \{e_\alpha \mid \exists\, \alpha \text{ such that } e_\beta \prec e_\alpha \text{ and } e_\alpha \in \mathcal{I}(e_\beta, r-1)\}$ It follows by induction that the cardinality of $\mathcal{I}(e_\beta, r)$ is finite for $0 \leq r$. For $r > \dim e_\beta$, $\mathcal{I}(e_\beta, r) = \emptyset$. The set of open cells e_α with $e_\alpha < e_\beta$ is the union over r of $\mathcal{I}(e_\beta, r)$ and this is finite.

Let E denote any set of points such that $E \cap e_\alpha$ has finite cardinality for each open cell α. We claim that the subspace topology inherited by E is the discrete topology. Let F be any subset of E. Note $F \cap \bar{e}_\beta \subset (F \cap e_\beta) \cup (\bigcup_{e_\alpha < e_\beta} F \cap e_\alpha)$ and since the right–hand side is a finite union of finite sets, the left–hand side is finite. Since X is Hausdorff, a finite set of points is closed so $F \cap \bar{e}_\beta$ is closed in $bare_\beta$. Hence F is a closed subset of X. Since F is an arbitrary subset of E, E inherits the discrete topology.

Finally, let $A \subset X$ be compact and let $E = \{x_\alpha\}$ be a choice of one point for each open cell e_α with $e_\alpha \cap A \neq \emptyset$. Then E inherits the discrete topology. But all the discrete subsets of compact sets are finite. Hence E is finite.

   (b) Show that A is contained in a finite subcomplex.

Proof: Let $K \subset X$ be the union of all open cells e_α where $e_\alpha < e_\beta$ for some e_β with $A \cap e_\beta \neq \emptyset$. By construction, K is a union of finitely many open cells. If $e_\alpha \subset K$ and $\dot{e}_\alpha \cap e_\beta \neq \emptyset$, then $e_\beta \prec e_\alpha$, so $e_\beta \subset K$. Hence K is a subcomplex.

3. For X a CW complex, X is compactly–generated.

Proof: We show $i_X\colon X_C \to X$ is a closed map. Let $A \subset X_C$ be closed. Then $A \cap D$ is a closed subset of D for all compact $D \subset X$. Since $\bar{e}_\alpha$ is the image of a compact set, it is compact, so $A \cap \bar{e}_\alpha$ is a closed subset of $\bar{e}_\alpha$ for all open cells α. But this means A is closed in X.