Math 608 Final
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Professor Taylor

When proving things during this test, you may assume earlier parts of a problem even if you can not do them. You may also assume known the homology of S^n, RP^n and CP^n as well as their cohomology rings.

Problem 1.

(a) Show CP^3 and $S^2 \times S^4$ have isomorphic homology. (You may just write down $H_*(CP^3)$, but you need to compute $H_*(S^2 \times S^4)$ using the Künneth formula.)

(b) Part (a) notwithstanding, show CP^3 and $S^2 \times S^4$ are not homotopy equivalent by showing that any map $f\colon S^2 \times S^4 \to CP^3$ has degree 0. Hint: Compute in cohomology.

(c) For good measure, show that any map $g\colon CP^3 \to S^2 \times S^4$ also has degree 0.

Problem 2. Suppose given an exact sequence $0 \to G_0 \to G_1 \to G_2 \to 0$.

(a) If A is an abelian group show

$$0 \to \mathrm{Hom}(A, G_0) \to \mathrm{Hom}(A, G_1) \to \mathrm{Hom}(A, G_2)$$

is exact.

(b) If A is a free abelian group show

$$0 \to \mathrm{Hom}(A, G_0) \to \mathrm{Hom}(A, G_1) \to \mathrm{Hom}(A, G_2) \to 0$$

is exact.

- Apply (b) to the groups $S_*(X)$. The sequence in (b) is natural so there results an exact sequence of chain complexes. When one applies homology, one gets a long exact sequence

$$\cdots \to H^r(X; G_0) \to H^r(X; G_1) \to H^r(X; G_2) \xrightarrow{\beta} H^{r+1}(X; G_0) \to \cdots .$$

The map β is again called the *Bockstein* associated to the exact sequence of coefficient groups.

(c) Compute the cohomology Bockstein sequences for $X = RP^2$ and coefficient sequence $0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \to 0$.

Problem 3. Let Y be a compact orientable manifold of dimension 7. Suppose we know the following groups: $H_7(Y; \mathbf{Z}) \cong \mathbf{Z}$, $H_6(Y; \mathbf{Z}) \cong \mathbf{Z}$, $H_5(Y; \mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$ and $H_4(Y; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$.

(a) Use the universal coefficients theorem to compute $H^r(Y; \mathbf{Z})$ for $r = 7$, 6, and 5.

(b) Recall the fact that all the homology groups of Y are finitely–generated plus the fact that $Ext(A, \mathbf{Z}) \cong T$ where $T \subset A$ denotes the torsion subgroup. Using these facts, what can you say about $H^4(Y; \mathbf{Z})$?

(c) Use Poincaré duality to compute $H_r(Y; \mathbf{Z})$ for $0 \le r < 3$.

Problem 4. Let M^m be a compact connected manifold of dimension $m > 2$ with boundary $\partial M = S^{m-1}$. If the map $\partial M \to M$ is null homotopic, show that $\widetilde{H}_*(M; \mathbf{Z}) = 0$.