Math 608 Midterm
April 3, 1998
Due 8am April 6

Problem 1. If X and Y are CW complexes, show that $X \times_C Y$ is also. Here $X \times_C Y = (X \times Y)_C$ as in problem 3, p. 213.

Problem 2. Let $f\colon [-1,1] \to [-1,1]$ be continuous and suppose $f(1) = \pm 1$, $f(-1) = \pm 1$. Show that

$$f_*\colon H_1([-1,1],\{-1,1\}) \to H_1([-1,1],\{-1,1\})$$

is given as follows:
a. multiplication by 1 if $f(-1) = -1$ and $f(1) = 1$;
b. multiplication by -1 if $f(-1) = 1$ and $f(1) = -1$;
c. multiplication by 0 if $f(1) = f(-1)$.

Problem 3. Let $z^n\colon S^1 \to S^1$ for $n \in \mathbf{Z}$ be the map obtained by considering S^1 as the unit circle in the complex plane and raising z to the n-th power. Choose cell structures on the range and domain S^1 so that z^n is cellular and use problem 2 to prove that the map $z^n_*\colon H_1(S^1) \to H_1(S^1)$ is multiplication by n.

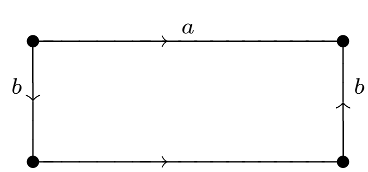
Problem 4. Let $f\colon S^m \to S^m$ be a map. If f is not onto, prove that $f_*\colon H_m(S^m) \to H_m(S^m)$ is the zero map.

Problem 5. Consider S^2 as the Riemann sphere. Equivalently, S^2 is the complex plane, $\mathbf{C}$ with the point at infinity added and topologized as the one point compactifiaction. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a degree n polynomial with all $a_i \in \mathbf{C}$. The polynomial p induces a map $p\colon \mathbf{C} \to \mathbf{C}$. Show that p extends to a map $p\colon S^2 \to S^2$ and further show that any two such maps of the same degree are homotopic.

Problem 6. Use the Mayer–Vietoris Theorem and problem 3 to show that $z^n\colon S^2 \to S^2$ induces the map $z^n_*\colon S^2 \to S^2$ which is multiplication by n.

Remark: Problems 4, 5 and 6 prove the Fundamental Theorem of Algebra: any degree n polynomial with coefficients in $\mathbf{C}$ has a root in $\mathbf{C}$.

Problem 7. Let K^2 be the Klein bottle. This is a CW complex with one 0-cell, two 1-cells and one 2-cell. Let the rectangle, R, below be the evident CW complex with four 0-cells, four 1-cells and one 2-cell. There is a cellular map $f\colon R \to K$ which takes all the vertices of R to the vertex of K; the four 1-cells of R to the two 1-cells of K as indicated by the arrows and labels; the 2-cell of R to the 2-cell of K by a homeomorphism on the interiors. Use this information to compute the boundary maps in the cellular chain complex for K. Problem 2 is useful here.



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