Final Exam., Math. 609, Fall, 2002

You will have 24 hours to work on this exam. You may use your notes. If the statement of some problem is unclear, please ask me about it.

1. Let T be an binary branching tree, consisting of finite sequences of 0's and 1's, closed under initial segments. A path through T is a function $\pi \in 2^\omega$ such that for all n, $\pi|n$ is in T. Consider the propositional language $S = \{P_n : n \in \omega\}$. For each $\sigma \in T$, let φ_σ be the conjunction of those P_n such that $\sigma(n) = 1$ and the negation of those P_n such that $\sigma(n) = 0$. For each k, there are only finitely many sequences σ of length k in T. Let ψ_k be the disjunction of φ_σ over these σ. Let $\Gamma = \{\psi_k : k \in \omega\}$. Show that if T is infinite, then Γ is consistent. Conclude that T has a path.

2. Every Archimedean ordered field can be embedded in the ordered field of real numbers. Using this fact, show that if T is the set of elementary first order sentences true in all Archimedean ordered fields, then T has models which are not Archimedean.

3. Let T be the set of all sentences true in $(\omega, <)$. Show that T has a model with a subset having the order type of the rationals.

4. For a countable, complete elementary first order theory T, let $I(T, \aleph_0)$ be the number of isomorphism types of countable models of T.

(a) Give an example such that $I(T, \aleph_0) = 1$.

(b) Give an example such that $I(T, \aleph_0) = 4$.

(c) Give an example such that $I(T, \aleph_0) = 2^{\aleph_0}$.

[You do not need to give proofs—just describe the examples.]

5. Let T be the theory of non-trivial vector spaces over the rationals.

(a) Describe (in terms of dimension) the prime model of T.

(b) Describe (again in terms of dimension) the countable saturated model of T.

6. Recall that the Kleene T-predicate is primitive recursive—$T(e, x, c)$ iff c is a halting computation for $\varphi_e(x)$. Using this, explain why, for a and e such that $a \in W_e$, W_e is the range of a primitive recursive function.

7. Let $I = \{e : W_e \neq \emptyset\}$.

(a) Show that I is c.e.

(b) Show that I is not computable.

8. Show that there is a non-computable set A whose jump, A' is computable relative to K. You may wish to use the following outline.

R_2e: $\varphi_e \neq \chi_A$.

R_2e+1: Put e into A', if possible.

The construction proceeds in stages. At stage $s+1$, you have determined a finite initial segment σ_s of χ_A, such that σ_s has length at least s, and guarantees satisfaction of R_k, for all $k < s$, and you want to define σ_{s+1} to guarantee satisfaction of R_s. The sequence $(\sigma_s)_{s\in\omega}$ should be computable relative to K.