Mathematics 609 — Final Exam – Version II

Answer all the following questions. (Question 1 has changed from Version I.) You may use any source that you wish as long as it isn't living. The test is due at 3:00 PM on Wednesday, December 17. Please give it to me personally or put it in my box on in the mailroom on the third floor.

1. Suppose that $X = \{\varphi_i \mid i \in \omega\}$ is a set of $\mathcal{L}$-sentences and T a theory such that for every model $\mathcal{A}$ of T, there is $\varphi_i \in X$ such that $\mathcal{A} \models \varphi_i$. Show that there is a finite subset $\{\varphi_{i_1}, \ldots, \varphi_{i_n}\}$ of formulas from X such that $T \vdash \varphi_{i_1} \vee \cdots \vee \varphi_{i_n}$.

2. Suppose that $\mathcal{A}$ is a countably infinite $\mathcal{L}$-structure in a countable language $\mathcal{L}$.

   (a) Show that $\mathcal{A}$ is an atomic model of $T = \mathrm{TH}(\mathcal{A})$ if and only if for every finite $Y \subseteq A$, $\mathcal{A}_Y$ is an atomic model of $T' = \mathrm{TH}(\mathcal{A}_Y)$.

   (b) With the same notation of the previous part, show that T is $\aleph_0$-categorical iff T' is $\aleph_0$-categorical.

3. Show that if $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{C}$ then $\mathcal{A} \preceq \mathcal{C}$.

4. Suppose that f, g, h, and k are primitive recursive functions. Let p and q be defined by simultaneous recursion as follows.

$$\begin{aligned} p(x,0) &= f(x) \\ q(x,,0) &= g(x) \\ p(x,y+1) &= h(x,y,p(x,y),q(x,y)) \\ q(x,y+1) &= k(x,y,p(x,y),q(x,y)) \end{aligned}$$

   Show that p and q are primitive recursive.

5. Show that Tot $\equiv_1$ Inf.

6. Soare, page 32, 1.19.

7. Soare, page 32, 1.21.

8. (a) Soare, page 38, 3.8, part a. (Notation: if C and D are sets, $C \oplus D$ denotes the set $\{2x \mid x \in C\} \cup \{2x+1 \mid x \in D\}$.

   (b) Use part a to give an alternate proof to Rice's Theorem, namely that if A is an index set such that $A \neq \emptyset$ and $A \neq \omega$, then A is not computable.

9. Soare, page 40, 3.16.

10. Soare, page 43, 4.10. (The notation $\lambda x[g(x,y)]$ is notation for the function $h(x) = g(x,y)$; in other words, the function of x with y fixed.)