Mathematics 609 — Midterm Test

Answer all the following questions. You may use any source that you wish as long as it isn't living. The test is due at 5:00 PM on Wednesday, October 29.

1. Given a set of students and a set of classes, suppose that each student wants to take a finite number of classes and that each class has a finite enrollment limit. Suppose that each finite set of students can be accomodated. Show that they all can be accomodated. (This problem can be done using only results from propositional logic. Let $P_{i,j}$ be a propositional variable representing the fact that student i wants course j.)

2. Problem III.8.1

3. Problem IV.1.2

4. Let T be a theory with an infinite set Γ of axioms. Suppose that for every finite set $\Delta \subseteq \Gamma$, there is a sentence $\varphi \in T$ such that $\Delta \not\vdash \varphi$. (That is, no finite subset of Γ is a set of axioms for T.) Show that there is no finite set of axioms for T.

5. Let T be true arithmetic; that is let $T = \mathrm{TH}(\mathcal{N})$. In this problem you will show that T has uncountably many nonisomorphic countable models.

   (a) Given a prime number p, write down a formula $\varphi_p(v)$ in the language of arithmetic that defines the relation "v is divisible by the prime number p."

   (b) Let S be any set of prime numbers. Show that there is countable model $\mathcal{A}$ of T and $a \in A$ such that if $p \in S$ then $\mathcal{A} \models \varphi_p[a]$ and if $p \notin S$ then $\mathcal{A} \models \neg\varphi_p[a]$.

   (c) Show that there are uncountably many countable models of T. (Note that the number of sets S of prime numbers is uncountable. So, for example, a single countable model $\mathcal{A}$ will not satisfy the property of part b for every S.)

6. Suppose that T is a consistent theory in a countable language such that $n(T, \aleph_0) = \aleph_0$. Show that there is a sentence φ such that $T \cup \varphi$ is complete.

7. Let $\mathcal{L} = \{0, S\}$ be the language with a constant symbol 0 and a unary function symbol S. Consider the following set of axioms.

   - $\forall x \forall y (S(x) = S(y) \rightarrow x = y)$

- $\forall x(x \neq 0 \leftrightarrow \exists y(y = S(x)))$
- for each n, the sentence $\forall x(S^n(x) \neq x)$ where $S^n(x) = S(S(\ldots S(x)\ldots))$

Note that the natural numbers with the usual successor function and 0 satisfy these axioms.

(a) Show that T is not finitely axiomatizable.

(b) Show that T is not $\aleph_0$-categorical.

(c) Show that T is κ-categorical for all $\kappa > \aleph_0$.

8. Suppose that $\mathcal{A}_i$ are structures such that $\mathcal{A}_i \preceq \mathcal{A}_{i+1}$ for every $i \in N$. Show that there is a natural way to define a structure $\mathcal{A}$ so that the universe of $\mathcal{A}$ is the union of the sets A_i and that for this structure $\mathcal{A}$ we have $\mathcal{A}_i \preceq \mathcal{A}$ for all i.

9. Show that the theory of dense linear orderings **with** endpoints is complete.

10. Suppose that $\{\varphi_i \mid i \in N\}$ is a set of $\mathcal{L}$-sentences and T a theory such that for each i there is a model $\mathcal{A}_i$ of T such that $\mathcal{A}_i \models \varphi_i$. Show that there is a finite subset $\{\varphi_{i_1}, \ldots, \varphi_{i_n}\}$ of these formulas such that $T \vdash \varphi_{i_1} \vee \cdots \vee \varphi_{i_n}$.