Final Exam

Math 609

Answer the following questions as clearly and completely as humanly possible.

1. State and prove the Soundness Theorem of propositional logic.
2. Let $\mathcal{L}$ be a first-order language. Give the definition of a term of $\mathcal{L}$. Give the definition of an atomic formula of $\mathcal{L}$. Give an example of a language $\mathcal{L}$ and an atomic formula in $\mathcal{L}$ which is a sentence.
3. State the Łoś-Vaught Test.
4. Let F be an infinite field and denote by $\mathrm{VS}(F)$ the theory of vector spaces over F in the language of vector spaces over F. Prove that this theory is complete.
5. Let $\mathcal{L} = (f)$ be the first-order language whose signature consists of just one unary function symbol f. Write a sentence φ (respectively, ψ) that expresses that f is injective (respectively, surjective).
6. Let $\mathcal{L}$ be as in the previous exercise and consider the theory

$$\{\varphi \wedge \psi, \forall x\,(f(x) \neq x), \forall x\,(f^2(x) \neq x), \ldots, \forall x\,(f^n(x) \neq x), \ldots\}.$$

   Prove that this theory is consistent, complete and not finitely axiomatizable.
7. Express in the language $\mathcal{L} = (<)$ the axioms for a dense linear order without endpoints. Prove that this theory is $\aleph_0$-categorical.
8. Let $f : N \to N$ be a total function on the natural numbers whose graph is definable in the language of arithmetic by a Σ_1 formula. Prove that the graph of f is then definable by a Δ_1 formula.
9. Give a definition by primitive recursion of the function

$$< x, y >= \frac{1}{2}(x+y)(x+y+1)+x.$$