Problem Set I

Math 609

1. Let φ be a sentence of the propositional language $\mathcal{S}$ and denote by $\ell(\varphi)$ (respectively, $r(\varphi)$) the number of left (respectively, right) parentheses in φ. Prove that for all sentences φ, $\ell(\varphi) = r(\varphi)$.

2. Let φ be a sentence of length n. For each k such that $1 \leq k \leq n$, let $\ell(\varphi, k)$ (respectively, $r(\varphi, k)$) denote the number of left (respectively, right) parentheses among the first k symbols of φ. Prove that for all k such that $1 \leq k < n$, $\ell(\varphi, k) > r(\varphi, k)$.

3. Let $\mathcal{S}$ be a propositional language. Define S_k for every natural number k by recursion as follows:
   i. $S_0 = \mathcal{S}$,
   ii. $S_{n+1} = S_n \cup \{(\neg\varphi) : \varphi \in S_n\} \cup \{(\varphi \wedge \psi) : \varphi, \psi \in S_n\}$.
   Prove that $\mathrm{Sent}(\mathcal{S}) = \bigcup_k S_k$.

4. A sentence φ of $\mathcal{S}$ is called a *literal* if it is either a sentence symbol p or the negation $(\neg p)$ of a sentence symbol. Show that every sentence φ of $\mathcal{S}$ is equivalent (semantically) to a finite disjunction of sentences each of which is a finite conjuction of literals. This is called the *disjunctive normal form* of φ. Prove also that φ is equivalent to a finite conjunction of sentences each of which is a finite disjunction of literals.

5. A connective c is a function $c : \{t, f\}^n \to \{t, f\}$. A set of connectives C is said to be *adequate* for propositional logic if every connective may be represented in the propositional whose connectives are given by C. Show that $\{\wedge, \neg\}$ are adequate for propositional logic. Give an example of a binary connective c such that the singleton set $\{c\}$ is adequate for propositional logic.

6. Show, by giving a deduction, that $\{\neg p, p \vee q\} \vdash q$.

7. A subset $\Sigma \subseteq \mathrm{Sent}(\mathcal{S})$ is *complete* if for every sentence σ of $\mathcal{S}$, exactly one of $\Sigma \vdash \sigma$ and $\Sigma \vdash \neg\sigma$ holds. Show that for any set Σ of sentences the following are equivalent:
   - The set of consequences of Σ is maximal consistent.
   - The theory Σ is complete.
   - The theory Σ has exactly one model.

8. **Interpolation theorem.** Assume that $\varphi \models \psi$. Show that either (i) φ is refutable, (ii) ψ is valid or (iii) there exists a sentence θ such that $\varphi \models \theta$ and $\theta \models \psi$ and every sentence symbol that occurs in θ occurs in both φ and ψ.

9. Prove the following. (In the latter two, assume that $\mathcal{S}$ is countable)
   - For every finite set K of models, there is a set Σ of sentences such that K is the set of all models of Σ.
   - Give an example of a set Σ of sentences such that the set of models of Σ is countably infinite.
   - Give an example of a countable set of models which cannot be represented as the set of models of some set Σ of sentences.