Problem Set II
Math 609

1. Given an interpretation $\mathcal{I} = (\mathcal{A}, h)$ and $\varphi,\ \psi \in \mathrm{Form}(\mathcal{L})$ prove the following.

   $\mathcal{I} \models \forall x_i\, \varphi$ if and only if for all $a \in A$, $\mathcal{I}_a^{x_i} \models \varphi$.

   $\mathcal{I} \models \varphi \vee \psi$ if and only if $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$.

   $\mathcal{I} \models \varphi \rightarrow \psi$ if and only if whenever $\mathcal{I} \models \varphi$, then $\mathcal{I} \models \psi$.

   $\mathcal{I} \models \varphi \leftrightarrow \psi$ if and only if ($\mathcal{I} \models \varphi$ iff $\mathcal{I} \models \psi$).

2. Give an example of formulas φ and ψ and an $\mathcal{L}$-structure M, such that $M \models \varphi \vee \psi$, but neither $M \models \varphi$ nor $M \models \psi$ holds.

3. Give a formal definition of the set $\mathrm{var}(\varphi)$ of free variables occuring in the formula $\varphi \in \mathrm{Form}(\mathcal{L})$, (that is, give a definition based on induction on the complexity of φ).

4. Let $\varphi \in \mathrm{Form}(\mathcal{L})$ with $\mathrm{var}(\varphi) = \{x_0, \ldots x_n\}$. Show that for an $\mathcal{L}$-structure $\mathcal{A}$, $\mathcal{A} \models \varphi$ if and only if $\mathcal{A} \models \forall x_0, \ldots x_n\ \varphi$.

5. Let $\mathcal{L} = \mathcal{L}_{\mathrm{PO}} = (\leq)$ be the first order language for a partial order. Consider the $\mathcal{L}$-structures $\mathcal{A} = (\mathcal{N}, \leq)$ where $\mathcal{N}$ denotes the natural numbers and $\leq$ the standard patial order on $\mathcal{N}$ and $\mathcal{B} = (\mathcal{N}, |)$ where $a|b$ denotes the binary relation "a divides b." Let $\sigma(x_0, x_1)$ be the formula

   $$x_0 \leq x_1 \wedge x_0 \neq x_1 \wedge \forall x_2 (x_0 \leq x_2 \wedge x_2 \leq x_1 \rightarrow x_0 = x_2 \vee x_1 = x_2),$$

   let $\varphi(x_0)$ be the formula $\exists x_1\, \sigma(x_0, x_1)$ and
   let $\psi(x_0)$ be the formula $\varphi(x_0) \wedge \forall x_1, x_2 (\sigma(x_0, x_1) \wedge \sigma(x_0, x_2) \rightarrow x_1 = x_2)$.
   Let h be any instantiation for $\mathcal{A}$ (respectively, $\mathcal{B}$) such that $h(x_0) \neq 0$. Evaluate φ and ψ in h. (Hint: Because $h(x_0) \neq 0$, the answer will not depend on h.)

6. Let $\mathcal{L}$ be a first order language with variables $x_0, x_1, \ldots, x_n, \ldots$. Prove that every instantiation $h : \mathrm{Var}(\mathcal{L}) \rightarrow \mathcal{A}$ extends uniquely to an $\mathcal{L}$-homomorphism h' from the term algebra of $\mathcal{L}$ with basis $\mathrm{Var}(\mathcal{L})$ to the $\mathcal{L}$-structure $\mathcal{A}$.

7. Let n be a natural number. Write a sentence φ_n in the language of pure equality such that for every set M, $M \models \varphi$ if and only if M has exactly n elements. Give an example of a sentence σ in a first-order language $\mathcal{L}$ such that whenever $M \models \sigma$, then M is infinite.