Problem Set III

Math 609

1. Give deductions for the axioms for equality.
   - $\vdash x \doteq x$
   - $\vdash x \doteq y \rightarrow y \doteq x$
   - $\vdash x \doteq y \wedge y \doteq z \rightarrow x \doteq z$

2. Give deductions for the following.
   - $\forall x \forall y\, \varphi \vdash \forall y \forall x\, \varphi$
   - $R(x) \vdash \exists y R(y)$

3. (Prenex normal form.) A formula φ is a *prenex formula* if it has the form $Q_0 x_0\, Q_1 x_1 \,\ldots\, Q_n x_n\, \psi$ where Q_i is $\exists$ or $\forall$ and ψ is quantifier-free. Prove that for any formula σ there is a prenex formula φ such that $\vdash \sigma \leftrightarrow \varphi$.

4. Define rigorously, by recursion on the formula $\varphi(x)$, when the term t is substitutable for x in φ.

5. Let $\mathcal{L}$ be a first-order language and $\{A_i : i \in I\}$ a set of $\mathcal{L}$-structures. Define the $\mathcal{L}$-structure $\prod_{i \in I} A_i$, the Cartesian product of the A_i.

6. Classify all the finite structures (up to isomorphism) in the following cases.
   - The $\mathcal{L}$-structures where $\mathcal{L} = \mathcal{L}_=$ is the language of pure equality.
   - The $\mathcal{L}$-structures where $\mathcal{L} = (R)$ and R is a unary relation symbol.
   - The $\mathcal{L}$-structures that satisfy the formula

     $$E(x,x) \wedge (E(x,y) \rightarrow E(y,x)) \wedge (E(x,y) \wedge E(y,z) \rightarrow E(x,z))$$

     in the language $\mathcal{L} = (E)$ where E is a binary relation symbol.
   - The $\mathcal{L}$-structures that satisfy the formula

     $$(f(x) = f(y) \rightarrow x = y) \wedge \forall y \exists x\, (f(x) = y)$$

     in the language $\mathcal{L} = (f)$ where f is a unary function symbol.