Problem Set IV
Math 609

The point of this problem set is to give a proof of the Compactness theorem without any hypotheses on the cardinality of the language.

Let I be a set. A set $\mathcal{F}$ of subsets of I is called a *filter* on I if it satisfies the following three conditions.

i. $\emptyset \notin \mathcal{F}$.

ii. If A, $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

iii. If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.

1. Let $A \subseteq I$. Verify that the set $\mathcal{F}_A = \{B \subseteq I : B \supseteq A\}$ is a filter. Filters of the form $\mathcal{F}_A$, $i \in I$, are called *principal.*

2. A subset $A \subseteq I$ is called *cofinite* if the complement $I \setminus A$ is finite. Verify that the set of all cofinite subsets of an infinite set I is a filter. It is called the Fréchet filter.

3. A set $\mathcal{G}$ of subsets of I is said to have the *finite intersection property* (FIP) if the intersection of any finite set of elements from $\mathcal{G}$ is nonempty. Prove that any set of subsets of I with the FIP is contained in a filter $\mathcal{F}$ on I and that there exists a unique smallest (with respect to inclusion) filter $\mathcal{F}$ that contains $\mathcal{G}$. This filter $\mathcal{F}$ is called the filter *generated by* $\mathcal{G}$.

4. Let S be an infinite set and $I = \mathcal{P}_\omega(S)$ the set of all finite subsets of S. For $i \in I$, define $i^* := \{j \in I : j \supseteq i\}$. Show that the set $\{i^* : i \in I\}$ has the FIP.

5. Prove that every filter is contained in a filter that is maximal with respect to inclusion. Maximal filters are called *ultrafilters.*

6. Let $\mathcal{F}$ be an ultrafilter. Prove that for every subset $A \subseteq I$, $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$.

7. Show that every principal ultrafilter has the form $\mathcal{F}_{\{i\}}$ for some element $i \in I$.

8. Let $A_1, A_2, \ldots, A_n$ be a finite collection of subsets of I such that the union $A_1 \cup A_2 \cup \ldots \cup A_n = I$. If $\mathcal{F}$ is an ultrafilter, show that one of the A_i belongs to $\mathcal{F}$.

9. Prove that an ultrafilter on an inifinite set I that is not principal contains the Fréchet filter.

10. Let $\mathcal{L}$ be a first-order language and $\mathcal{F}$ be an ultrafilter on I. For a set $\{A_i : i \in I\}$ of $\mathcal{L}$-structures, define the relation $a \equiv_{\mathcal{F}} b$ on the cartesian product $\prod_{i \in I} A_i$ by
$$a \equiv_{\mathcal{F}} b \text{ iff } \{i : a_i = b_i\} \in \mathcal{F}.$$
We say that a and b are equal almost everywhere (with respect to $\mathcal{F}$). Prove that the relation $a \equiv_{\mathcal{F}} b$ is an equivalence relation. The class of a modulo this relation is denoted $a/\mathcal{F}$.

11. Denote by $\prod_i A_i/\mathcal{F}$ the set of equivalence classes modulo the relation $\equiv_{\mathcal{F}}$. It is called the *ultra-product* of the A_i with respect to $\mathcal{F}$. Define an $\mathcal{L}$-structure on $\prod_i A_i/\mathcal{F}$.

12. Prove Łoś's Theorem. For every formula $\varphi(x_0, x_1, \ldots, x_n)$ of $\mathcal{L}$ in the free variables $x_0, x_1, \ldots, x_n$ and $a_0, a_1, \ldots, a_n \in \prod_i A_i$,
$$\prod_i A_i/\mathcal{F} \models \varphi[a_0/\mathcal{F}, a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}] \text{ iff } A_i \models \varphi[(a_0)_i, (a_1)_i, \ldots, (a_n)_i] \text{ for almost all } i.$$

13. Use Łoś's Theorem (and exercise 4) to prove that every finitely satisfiable set Σ of formulas of $\mathcal{L}$ is satisfiable.