Problem Set IX - Clubs
Math 609

Definition. Let μ be a limit ordinal. A set $C \subseteq \mu$ is *closed* in μ if for every limit ordinal $\delta < \mu$, whenever $C \cap \delta$ is unbounded in δ, then $\delta \in C$.

1. Let μ be a limit ordinal. Prove that $C \subseteq \mu$ is closed in μ if and only if it is a closed subset of μ with respect to the order topology.

Definition. Let μ be a limit ordinal. A set $C \subseteq \mu$ is a *club* of μ if it is closed and unbounded in μ.

2. Let μ be an ordinal such that $\mathrm{cf}(\mu) > \omega$. Show that if A and B are clubs of μ, then so is $A \cap B$.

Definition. Let $\mathrm{Club}(\mu)$ denote the filter on μ generated by the clubs of μ, i.e., $\mathrm{Club}(\mu) := \{X \subseteq \mu : X \supseteq C \text{ for some club of } \mu\}$.

3. Let μ be an ordinal such that $\mathrm{cf}(\mu) > \kappa$ and let $\{X_\alpha : \alpha < \kappa\} \subseteq \mathrm{Club}(\mu)$. Verify that the intersection $\bigcap_{\alpha<\kappa} X_\alpha$ belongs to $\mathrm{Club}(\mu)$.

Definition. A subset $X \subseteq \mu$ is *stationary* in μ if $X \cap C \neq 0$ for every club C of μ.

4. Let λ be a regular cardinal and $\mathrm{cf}(\mu) > \lambda$. Show that the set $\{\gamma < \mu : \mathrm{cf}(\gamma) = \lambda\}$ is stationary in μ.

5. Let $\kappa > \omega$ be regular and suppose that for each $\alpha < \kappa$, C_α is a club of κ. Prove that the *diagonal intersection* D is also a club of κ where $D := \{\gamma : \forall\, \alpha < \gamma\, (\gamma \in C_\alpha)\}$.

6. Prove Fodor's Lemma: Let $\kappa > \omega$ be a regular cardinal and $S \subseteq \kappa$ stationary. If $f : S \to \kappa$ is a function such that for all γ in S, $f(\gamma) < \gamma$, then for some $\alpha < \kappa$, $f^{-1}(\{\alpha\})$ is stationary in κ. (Hint: Suppose not and apply 5.)

Definition. A $\diamond$-sequence is a sequence $\{A_\alpha : \alpha < \omega_1\}$ of sets $A_\alpha \subseteq \omega_1$ such that for every $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary.

7. Let $\diamond$ be the prediction principle: There exists a $\diamond$-sequence. Show that $\diamond$ implies CH.