Problem Set V
Math 609

1. Prove that if a theory T in a first-order language $\mathcal{L}$ has arbitrarily large finite models, then it has an infinite model.

2. Let $\mathcal{F} = (F, +, \cdot\,; 0, 1)$ be field and consider the curve in F^3 whose elements are the points of the form (t, t^2, t^3) as t ranges over F. This curve is definable by the formula $\exists t\,(x \doteq t \wedge y \doteq t^2 \wedge z \doteq t^3)$. This is the *parametric* definition of the curve. Find a quantifier-free formula $\varphi(x, y, z)$ that defines the same curve in F^3.

3. Let F be a field. The language for vector spaces over F is the language $\mathcal{L} = (+, -; 0) \cup \{a\}_{a \in F}$ where the three symbols $+$, $-$ and 0 are intended to interpret the underlying abelian group structure of the vector space and for every element $a \in F$, the unary function symbol a is intended to interpret the action (scalar multiplication) of a on the undelying abelian group. The theory $\mathrm{VS}(F)$ of vector spaces over F is axiomatized by the axioms for an abelian group together with the following four axiom schemata:

   - $1 \cdot x \doteq x$.
   - For all $c \in F$, $c(x + y) \doteq c(x) + c(y)$.
   - For all c, $d \in F$, $(c + d)(x) \doteq c(x) + d(x)$.
   - For all c, $d \in F$, $c(d(x)) \doteq (cd)(x)$.

   Let $A = (a_{ij})$ be an $m \times n$ matrix with entries from F and abbreviate by $A\,\mathbf{x} \doteq \mathbf{y}$ the formula in the free variables $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots y_m)$ that defines multiplication by A from the left on a vector space over F. Write out the formula $A\,\mathbf{x} \doteq \mathbf{y}$ explicitly in the language for vector spaces over F and find a quantifier-free formula $\varphi(\mathbf{y})$ such that

   $$\mathrm{VS}(F) \vdash \varphi(\mathbf{y}) \leftrightarrow \exists\, \mathbf{x}\; A\,\mathbf{x} \doteq \mathbf{y}\,.$$

4. Prove that the relation $<$ is definable in the ring $(R, +, \cdot\,; 0, 1)$ of real numbers.

5. Let $\mathcal{L} = (+, \cdot\,; 0, 1)$ and consider the ring of real numbers as an $\mathcal{L}$-structure. If $A \subseteq R$, an element $c \in R$ is called *algebraic* (respectively, *definable*) over A if there is a formula $\varphi(x, a_0, a_1, \ldots, a_n)$ with parameters from A such that $(R, +, \cdot\,; 0, 1) \models \varphi(c, a_0, a_1, \ldots, a_n)$ and $\varphi(R, a_0, a_1, \ldots, a_n)$ is finite (respectively, singleton). Show that every element $c \in R$ that is algebraic over A is definable over A.

6. Overspill. Let $\mathcal{A} \models PA$ be a nonstandard model of Peano Arithmetic and $\varphi(x, a_0, a_1, \ldots, a_n)$ a formula in the language for arithmetic with parameters in $\mathcal{A}$. Show that if $\varphi(\mathcal{A}, a_0, a_1, \ldots, a_n)$ contains arbitrarily large standard elements, then it contains a nonstandard element.

7. Let $\mathcal{A} \models PA$ be a model of Peano Arithmetic. Define the equivalence relation $a \sim b$ if $|a - b| \in N$. Show that if $\mathcal{A}$ is nonstandard, there is an equivalence class without a prime representative.

8. Show that the relation $<$ is definable in the ring $(Z, +, \cdot\,, 0, 1)$ of integers.