Problem Set VI
Math 609

1. A structure $\mathcal{F} = (F, P; +, \cdot; 0, 1)$ is an *ordered field* if it satisfies the axioms for a field (in the reduct $(+, \cdot; 0, 1)$) and the following axioms involving the unary relation, that is, predicate, symbol P. This symbol is intended to interpret the positive elements of the ordered field $\mathcal{F}$.

   OF1. $P(x) \wedge P(y) \rightarrow P(x + y)$.

   OF2. $P(x) \wedge P(y) \rightarrow P(x \cdot y)$.

   OF3. $x \neq 0 \rightarrow (P(x) \vee P(-x))$

   OF4. $\neg(P(x) \wedge P(-x))$

   Show that every ordered field has characteristic 0, that is, for every nonzero natural number n, $OF \vdash nx \doteq 0 \rightarrow x \doteq 0$. Prove that the binary relation $<$ on the structure $\mathcal{F} = (F, P; +, \cdot; 0, 1)$ defined by $x < y = P(y - x)$ is a dense linear order without endpoints.

2. An ordered field $\mathcal{F} = (F, P; +, \cdot; 0, 1)$ is called *Archimidean* if for every $a \in F$, there is a natural number n such that $|a| < n$. Prove the following.

   - Every ordered field is elementarily equivalent to a non-Archimedean field.
   - The set of elements $R = \{a \in F : |a| < n \text{ for some natural number } n\}$ is a subring of $(F, +, \cdot; 0, 1)$
   - The set of non-units of R is the ideal of infinitesimal elements

   $$I := \{a : |a| < 1/n \text{ for some positive natural number } n\}.$$

   - R/I is an Archimedean field.

   In the remaining exercises, assume that $\mathcal{L}$ is a countable language and that T is a complete theory in $\mathcal{L}$.

3. Prove that if a model $\mathcal{A} \models T$ is atomic and countable, then it is the prime model of T.

   Definition. A countable model $\mathcal{A} \models T$ is *homogeneous* if given $a, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$ such that $\operatorname{tp}_{\mathcal{A}}(a_1, \ldots, a_n) = \operatorname{tp}_{\mathcal{A}}(b_1, \ldots, b_n)$, there is a $b \in \mathcal{A}$ such that $\operatorname{tp}_{\mathcal{A}}(a, a_1, \ldots, a_n) = \operatorname{tp}_{\mathcal{A}}(b, b_1, \ldots, b_n)$.

4. Verify that if $\mathcal{A} \models T$ is countable and atomic, then it is homogeneous.

5. Verify that if $\mathcal{A} \models T$ is countable and saturated, then it is homogeneous.

6. Prove that if $\mathcal{A} \models T$ is homogeneous and $a_1, \ldots, a_n, b_n, \ldots, b_n \in \mathcal{A}$ are such that $\operatorname{tp}_{\mathcal{A}}(a_1, \ldots, a_n) = \operatorname{tp}_{\mathcal{A}}(b_1, \ldots, b_n)$, then there is an automorphism $\eta : \mathcal{A} \to \mathcal{A}$ such that $\eta : a_i \mapsto b_i$.

7. Prove that if $\mathcal{A}$, $\mathcal{B} \models T$ are countable homogeneous models of T, then $\mathcal{A} \cong \mathcal{B}$ if and only if they realize the same n-types for all $n < \omega$.