Problem Set VII - The Midterm
Math 609

Answer the following questions as clearly and completely as you can. In solving a given exercise, you are free to assume (and encouraged to cite) any of the previous exercises.

1. Let $\mathcal{L} = (E)$ be a first-order language where E is a binary relation symbol. What is an $\mathcal{L}$-structure?

2. Let $\mathcal{L} = (E)$ be as above and let $\mathcal{A} = (A, E^{\mathcal{A}})$ and $\mathcal{B} = (B, E^{\mathcal{B}})$ be $\mathcal{L}$-structures. What is an $\mathcal{L}$-homomorphism $\eta : \mathcal{A} \to \mathcal{B}$?

3. Express the axioms for an equivalence relation in the language $\mathcal{L} = (E)$. Denote the theory of an equivalence relation, that is, the set of consequences of these axioms, by T_{eq}.

4. Calculate $I(T_{\text{eq}}, n)$ for $n = 1$, 2, 3 and 4.

5. Let $\mathcal{A} = (A, E^{\mathcal{A}})$ be an $\mathcal{L}$-structure such that $\mathcal{A} \models T_{\text{eq}}$. For $a \in A$, denote the equivalence class of a modulo $E^{\mathcal{A}}$ by $a/E^{\mathcal{A}}$ and denote by $A/E^{\mathcal{A}}$ the set of equivalence classes of A modulo $E^{\mathcal{A}}$. Prove that if $\mathcal{B} = (B, E^{\mathcal{B}}) \models T_{\text{eq}}$, then an $\mathcal{L}$-homomorphism $\eta : \mathcal{A} \to \mathcal{B}$ induces a function $\eta/E : A/E^{\mathcal{A}} \to B/E^{\mathcal{B}}$ in the natural way.

6. Prove that an $\mathcal{L}$-homomorphism $\eta : \mathcal{A} \to \mathcal{B}$ is an embedding if and only if both η and η/E are injective.

7. Let n be a natural number and $\mathcal{A} \models T_{\text{eq}}$. Exhibit a formula $P_n(x)$ that defines in $\mathcal{A}$ the set of all $a \in A$ such that $|a/E^{\mathcal{A}}| = n$.

Assume from now on that $T \supseteq T_{\text{eq}}$ is a complete theory.

8. Show that if $T \models \exists x\, P_n(x)$, then the formula $P_n(x)$ isolates a type in $S_1(T)$, that is, the basic open subset $[P_n(x)]$ of $S_1(T)$ is singleton.

9. Consider the 1-type $p_\infty(x) := \{\neg P_n(x) : n < \omega\}$. Prove that if p_∞ is T-consistent, then it determines a unique point of $S_1(T)$.

10. Suppose that p_∞ is T-consistent. Prove that p_∞ is isolated in $S_1(T)$ if and only if there are only finitely many natural numbers n for which $T \models \exists x\, P_n(x)$.

11. Let $\mathcal{A} = (A, E^{\mathcal{A}})$ and $\mathcal{B} = (B, E^{\mathcal{B}})$ be *countable* $\mathcal{L}$-structures such that $\mathcal{A}, \mathcal{B} \models T_{\text{eq}}$. Show that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if for every natural number n, $|P_n(\mathcal{A})/E^{\mathcal{A}}| = |P_n(\mathcal{B})/E^{\mathcal{B}}|$ and also $|p_\infty(\mathcal{A})/E^{\mathcal{A}}| = |p_\infty(\mathcal{B})/E^{\mathcal{B}}|$.

12. Prove that T is $\aleph_0$-categorical if and only if it has an infinite model and the Stone space $S_1(T)$ is finite.

13. Prove that T is $\aleph_1$-categorical if and only if the following three conditions are satisfied:

a) $S_1(T)$ is finite.

b) The definable set $p(\mathcal{A})$ is infinite for exactly one of the 1-types $p \in S_1(T)$.

c) $|p_\infty(\mathcal{A})/E^{\mathcal{A}}| \leq 1$.