The Stone space of types
Math 609

Let $\mathcal{L}$ be a first-order language. An n-type $\Sigma = \Sigma(x_1, \ldots, x_n)$ is a deductively closed subset of Form($\mathcal{L}$) such that for every $\varphi \in \Sigma(x_1, \ldots, x_n)$, var($\varphi$) $\subseteq \{x_1, \ldots, x_n\}$. A theory is 0-type.

- An n-type $\Sigma(x_1, \ldots, x_n)$ is consistent if and only if for every finite subset $\varphi_1, \ldots, \varphi_m \in \Sigma$ there is an $\mathcal{L}$-structure $\mathcal{A}$ such that
$$\mathcal{A} \models \exists\, x_1, \ldots, x_n\, (\bigwedge_{i=1}^{m} \varphi_i).$$
By the Compactness Theorem, this is equivalent to the existence of an $\mathcal{L}$ structure $\mathcal{A}$ and $a_1, \ldots, a_n \in \mathcal{A}$ such that
$$\mathcal{A} \models \Sigma(a_1, \ldots, a_n).$$

- An n-type $p = p(x_1, \ldots, x_n)$ is complete if it is consistent and for every formula $\varphi \in$ Form($\mathcal{L}$), such that var(φ) $\subseteq \{x_1, \ldots, x_n\}$, either $\varphi \in p$ or $\neg\, \varphi \in p$.

  Example. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $a_1 \ldots, a_n \in \mathcal{A}$, then
$$\operatorname{tp}_{\mathcal{A}}(a_1 \ldots, a_n) := \{\varphi(x_1 \ldots, x_n) \mid \mathcal{A} \models \varphi(a_1 \ldots, a_n)\}$$
is complete. By the Compactness Theorem, every complete n-type is of this form.

- The Stone space $S_n^{\mathcal{L}}$ is the set whose points are the complete n-types in the language $\mathcal{L}$ and a basis of open subsets is given by the sets
$$[\varphi(x_1 \ldots, x_n)] := \{p \in S_n^{\mathcal{L}} \mid \varphi \in p\}.$$
Every open subset of $S_n^{\mathcal{L}}$ is of the form $\bigcup_{i \in I}[\varphi_i]$ and every closed subset is of the form $\bigcap_{i \in I}[\psi_i]$. By the Compactness Theorem, the topological space $S_n^{\mathcal{L}}$ is compact.

  Example. ($n = 0$.) The points of $S_0^{\mathcal{L}}$ are the complete theories of $\mathcal{L}$-structures.

- Let Σ be a consistent n-type. Define the $S_n^{\mathcal{L}}(\Sigma) \subseteq S_n^{\mathcal{L}}$ to be the closed subset $\bigcap_{\varphi \in \Sigma}[\varphi]$. Then $p \in S_n^{\mathcal{L}}(\Sigma)$ if and only if $p \supseteq \Sigma$ is a completion of Σ.

  Example. Let T be a theory in $\mathcal{L}$. Then $S_n^{\mathcal{L}}(T)$ denotes the closed subset of $S_n^{\mathcal{L}}$ of all n-types realized in some model of T.

- (constants vs. variables) Let $\mathcal{L}' = \mathcal{L}(c_1, \ldots c_m)$ be an expansion of $\mathcal{L}$ by m new constant symbols. If $m \leq n$, then the function $S_n^{\mathcal{L}} \to S_{n-m}^{\mathcal{L}'}$ defined by $p(x_1, \ldots, x_n) \mapsto p(c_1, \ldots, c_m, x_1, \ldots, x_{n-m})$ is a homeomorphism.

  Example. ($m = n$.) The Stone space $S_m^{\mathcal{L}}$ is homeomorphic via the evaluation map to the the Stone space $S_0^{\mathcal{L}'}$ of complete theories in the expanded language $\mathcal{L}' = \mathcal{L}(c_1, \ldots, c_m)$.