MATH 621: TOPICS IN ALGEBRAIC GEOMETRY

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References and sources:

(1) Migliore's book
(2) Pragmatic notes (Geramita and Migliore)
(3) Migliore's Kyoto paper

Outline

I. Graded modules over a polynomial ring
  A. Macaulay, CoCoA
  B. $R = k[X_0, ..., X_n]$, $\mathbb{P}^n$
    (i) Noetherian
  C. shift (and twist)
  D. free modules
  E. finitely generated modules
    (i) Examples
      (a) ideal
      (b) submodule of a free module (column space)
  F. module structure
    (i) picture of multiplication
    (ii) example: $k \oplus k$
    (iii) represent structure by matrices of linear forms
  G. minimal generators
  H. annihilator – homogeneous ideal
  I. finite length
    (i) diameter
    (ii) Buchsbaum index ($\leq$ diameter)

J. dual module

II. Homogeneous ideals and schemes

A. dictionary – see Cox-Little-O'Shea

(i) prime ideals, radical ideals, etc.

B. saturated ideals

(i) example: compare the ideals $I_1 = (x, y)$, $I_2 = (x, y^2)$, $I_3 = (x^2, xy, y^2)$, $I_4 = (x^2, y^2)$, $I_5 = (x^2, xy, xz, y^2, yz)$.

(ii) definition of saturated ideal, saturation

(iii) example from p. 2 of Fall 1992 – ideal of 4 points in $\mathbb{P}^3$

C. schemes

D. Hilbert function and polynomial

(i) degree, arithmetic genus

(ii) example: $\mathbb{P}^n$ has dimension n, degree 1

(iii) example: a hypersurface is defined by a homogeneous polynomial of degree d. It has dimension $n - 1$, degree d.

E. linear systems, base locus

F. Examples

(i) Fat points (I)

(a) Point of view: square the ideal and saturate. Geometrically, we're looking for curves that are singular at all the points.

(b) Example: three fat points lie ("unexpectedly") on a cubic, namely the union of the three lines that they span.

(c) When is the square already saturated? Example: one point, two points. We'll see with liaison that if it's a complete intersection then it's saturated.

(ii) double line

G. primary decomposition

III. minimal free resolutions – follow Tony's notes.

A. Goal: want the kernel to be free. The kernel is the module of i-th syzygies.

B. "relations on the relations"

C. homological dimension

D. graded Betti numbers

E. Hilbert Szyzgy theorem

F. Castelnuovo-Mumford regularity (I)

G. criterion for minimality (give proof of both directions)

H. examples

(i) first do $I = (x, y)$, then $I = (x, y, z)$

(ii) notice that it is a complex, but more precisely it is *exact*

(iii) do $I = (x^2, y^2, z^2)$ and then $I = (x^2, xy, y^2)$, first trying to mimic the Koszul resolution, then giving the correct resolution

(iv) resolution for k

I. Useful constructions

(i) add trivial summand

(ii) direct sum resolution

(a) Example: resolution for k^2

(b) Example: resolution for $k \oplus k$ with trivial and nontrivial structure

(iii) Koszul resolution for a regular sequence

J. Hom and Ext

(i) motivation: a resolution is a complex, hence composition of two maps is zero, hence composition of the transposes is also zero. Want to make some sense of this, by saying that if we go in the opposite direction we have at least a complex, and Ext will measure how far we are from exactness.

(ii) Examples

(a) $M = R(a) \Rightarrow \mathrm{Hom}_R(M, R) \cong R(-a)$

(b) M has finite length $\Rightarrow \mathrm{Hom}_R(M, R) = 0$.

(c) Look at some of the complete intersections we've looked at: self-dual!

IV. arithmetically Cohen-Macaulay schemes

A. Cohen-Macaulay rings

      (i) regular sequence
      (ii) depth
      (iii) dimension of a ring
   B. a zeroscheme is always ACM
   C. Auslander-Buchsbaum
      (i) examples (also of non-ACM rings)
   D. projective dimension = codimension
   E. Cohen-Macaulay type
   F. arithmetically Gorenstein (I)
      (i) complete intersection, Koszul resolution
      (ii) in codimension two CI $\Leftrightarrow$ arithmetically Gorenstein. In higher codimension this is not true.
         (a) Example: five points in $\mathbb{P}^3$. Describe why, from maximal rank point of view, we expect Gorenstein.
      (iii) We'll come back to it in much more detail later.
   G. Minimal Resolution Conjecture (idea is the same as the maximal rank idea that just gave us Gorenstein for 5 points in $\mathbb{P}^3$).
      (i) generic Hilbert function
   H. canonical module
      (i) resolution of the canonical module for ACM schemes
      (ii) duality for arithmetically Gorenstein schemes
      (iii) Prop 4.1.1, Cor 4.1.3 of my book
   I. Hilbert function of points
   J. truncation
      (i) basic lemma
      (ii) DGM – will talk in seminar
V. Sheaves and sheaf cohomology
   A. basics about sheafification– see PRAGMATIC notes
   B. Sheafification of a short exact sequence of graded modules

C. Sheafification of a resolution

VI. Schemes and Deficiency Modules

A. deficiency modules (definition)

B. Relation between Ext groups and cohomology of the ideal sheaf. (See Schwartau Lemma 31, then generalize it. See also BM4.)

(i) sketch proof for surface in $\mathbb{P}^5$.

(ii) Corollary: ACM $\Leftrightarrow$ deficiency modules are zero.

(iii) Give second proof without Ext's.

C. Examples

(i) two skew lines

(ii) disjoint union of a line and a conic

(iii) disjoint union of a line and a plane curve

(iv) maximal rank curve, e.g. general set of skew lines (Hartshorne-Hirschowitz), general rational curve (Hirschowitz). Mention Ballico-Ellia.

D. Castelnuovo-Mumford regularity (II)

(i) examples

VII. Hyperplane and Hypersurface sections

A. introduction: depth $R/IV = 1$ iff every $\bar{F}_2$ is a zero divisor in $R/(I, F_1)$ (for any choice of F_1 which is not a zero divisor in R/I_V), and this is true iff (I_V, F_1) is not saturated. So we want to study ideals of this form.

B. algebraic, geometric hyperplane or hypersurface section

C. exact sequences that arise – see Pragmatic notes and book

(i) Corollary: K_F measures the failure of $I_V + (F)$ to be saturated.

(ii) Corollary: $I_V + (F)$ is saturated iff $(M^1)(V) = 0$. (Use Serre's theorem, [Hartshorne] p. 228.)

D. Mention Dubreil result.

E. What's preserved under *geometric* hyperplane sections?

(i) dimension is one less

(ii) the ACM property is preserved.

(iii) In fact, the converse holds if stated properly. Note need dimension ≥ 2.

(a) Example of projection of Veronese.

(b) Say a few words about the curve case, mentioning Strano, Re, H-U, me.

F. Hilbert function, first difference

G. Artinian reduction of an ACM scheme

(i) Hilbert function is the first difference: h-vectors

H. graded Betti numbers preserved in aCM case

I. CoCoA, macaulay

VIII. Deficiency modules revisited

A. Note structure is important to specify the module, not just the dimension of the components. This is the "hard" part. Depends a lot on the geometry of the curve (or scheme).

B. Example: disjoint union in $\mathbb{P}^3$ of a line and a conic.

C. Philosophy (state in the case of curves, maybe higher too): those linear forms having "unusual" rank on the HR-module correspond to planes meeting the curve in "unusual" ways, either by containing a component of the curve or by having unusual postulation for the hyperplane section.

D. What can happen in negative degrees to $h^1(\mathcal{I}_C(t))$ and $h^2(\mathcal{I}_C(t))$?

(i) If C is reduced then

(a) $M(C)_i = 0$ for $i < 0$.

(b) $M(C)_0 =$ (number of connected components of C) - 1.

(ii) Rao module dimensions are non-decreasing.

(iii) In fact dimensions are strictly increasing.

E. General question: which shifts occur?

(i) Theorem: for any M, if you shift far enough to the right then curve exists. (Just state for $\mathbb{P}^3$.) Sketch proof.

(ii) Stress that it's not true without shifting.

(iii) Now turn to a related question, with direct sums.

IX. Liaison Addition

A. Remark about how all modules occur, so Phil's is a natural question.

B. Note first question is false: Take both curves to be disjoint union of two lines.

C. Schwartau's result (state and prove for curves in $\mathbb{P}^3$).

D. give geometric interpretation.

E. Generalized result (just state, and mention that the proof is very similar with a few small complications)

F. Application: Basic double linkage. (We'll see where the name comes from later.)

G. Basic double G-linkage (1st visit).

H. Application: construct examples of schemes with embedded components

I. Construct Buchsbaum curves

  (i) k^2, $k \oplus k$ (Berlin picture)

  (ii) mention both of these satisfy $\nu = 3N + 1$, $\alpha = 2N$.

X. Buchsbaum subschemes of $\mathbb{P}^n$, especially Buchsbaum curves in $\mathbb{P}^3$

A. Definition of Buchsbaum curve.

B. Definition of Buchsbaum subscheme.

C. Rao's result: relation between resolution of curve and module

D. See how much is forced on a curve by knowing that its module has trivial structure: BSV result, Amasaki results.

E. GM1: show what shifts can occur for Buchsbaum curves in $\mathbb{P}^3$.

F. connected except for 2 skew lines

XI. Gorenstein Schemes and Ideals

A. Recall definition

B. Constructions and Theorems

  (i) sums of G-linked ideals

  (ii) Linear system construction

  (iii) Sections of Buchsbaum-Rim sheaves of odd rank.

XII. Liaison: Definitions, Examples, Questions

A. history

B. *liaison* (start off in arbitrary codimension)— intuitive definition, geometric liaison, algebraic liaison, precise definition.

C. geometric link $\Rightarrow$ algebraic link, but not converse.

D. We'll see soon that algebraic link and no common component $\Rightarrow$ geometric link.

E. $I_X : I_V$ is saturated

F. even liaison, $\mathcal{L}$ = even liaison class

G. Questions:

  (i) Find connections between the linked schemes (degree, genus, and especially more subtle connections)

  (ii) Do geometric and algebraic links generate the same equivalence relation?

  (iii) Is this a trivial equivalence relation? Mention aCM example, especially in codimension two. Also the difference between codim 2 and the general case. (In particular, in $\mathbb{P}^2$ codimension 2 it *is* trivial, and in codimension 1 anywhere, but not otherwise.)

  (iv) Parameterize the (even) liaison classes— give necessary and sufficient conditions for two subschemes to be in the same (even) liaison class

  (v) Describe any one even liaison class— *structure*— in particular, are they all the same?

  (vi) Applications?

H. Examples

  (i) CoCoA

  (ii) twisted cubic $\sim$ line

  (iii) concocted example where residual is non-reduced

  (iv) mention that sum of degrees of linked schemes = degree of the complete intersection

  (v) rational sextic with a 5-secant line— linking with two cubics *forces* the residual to be non-reduced. (Need to show that there's exactly one 4-secant in addition to the 5-secant.)

  (vi) self-linked

XIII. First results (arbitrary codimension)

A. Any two complete intersections (of the same dimension) are linked— for example see Phil's Theorem 14.
   (i) definitions of licci, glicci
   (ii) Casanellas- Miró-Roig result for gorensteins being glicci
B. linearly equivalent $\Rightarrow$ evenly linked in two steps
C. Choose V_1 and $X \Rightarrow$ get $V_2 \overset{X}{\sim} V_1$
D. For curves, non locally CM $\Rightarrow$ singleton class. So restrict to locally CM.
E. If $V_1 \overset{X}{\sim} V_2$ then V_1, V_2 are equidimensional of the same dimension, and without embedded components (see Phil p. 28).
F. Linked scheme is again locally CM of the same dimension (Phil Theorem 15) – follow my book prop. 5.2.2, prop. 5.2.3.
G. dualizing sheaf exact sequence
H. Mapping Cone
   (i) Corollary: for aCM codimension two, linking using two (resp. one, zero) minimal generators drops the number of minimal generators for the residual by one (resp. leaves it the same, increases it by one). (Mention Hilbert-Burch.)
I. degree, genus, Hilbert function of residual
J. Examples
   (i) twisted cubic $\sim$ line again
   (ii) two things of same degree linked $\Rightarrow$ same genus
K. Are hyperplane sections linked? Yes.
L. Necessary conditions— Hartshorne-Schenzel
   (i) Note that for dimension ≥ 2, the *configuration* of modules is preserved up to shift.
   (ii) Useful for showing things are *not* linked (as in next "chapter")
   (iii) Corollary: necessary condition for self-linked
   (iv) Corollary: the property of being Buchsbaum or aCM is preserved under liaison.
M. Theorem: Apéry-Gaeta-Peskine-Szpiró. (codimension two) The part that's a corollary is that *licci* $\Rightarrow$ arithmetically CM. For the other direction, use the

fact that linking using minimal generators drops the number of generators. For this it'll be necessary to talk about Hilbert-Burch stuff

N. up to shift, all modules occur (Schenzel, Rao)

O. see why even liaison is better

P. Minimal shift

  (i) Lemma: any rightward shift occurs

  (ii) Leftward shifts in general

Q. Special case: Curves

  (i) Recall what can happen in negative degrees to $h^1(\mathcal{I}_C(t))$ (non-decreasing)

  (ii) Minimal shift exists (and all subsequent shifts occur)

R. In general: notion of shift (partition class), minimal shift, notation $\mathcal{L}^h$

XIV. Geometric Invariants of Liaison (especially curves in $\mathbb{P}^3$)

A. How do you describe the module structure? $\phi_d : S_1 \to \text{Hom}(M_d, M_{d+1})$, then describe it in terms of matrices.

  (i) Example– diameter two, dimensions 1,1. Discuss the possible isomorphisms from this point of view.

  (ii) Mention Ballico-Bolondi, $\nu_{\mathbf{P}^n}(m_1, ..., m_t)$

  (iii) Example– diameter 3, dimensions 1,1,1. Possible isomorphisms.

B. Definitions of the degeneracy loci

  (i) Porteous' Formula

C. Some results about when you know that a linear form corresponds to a point of the degeneracy locus.

  (i) standard result (postulation of hyperplane section)

  (ii) some cases when the hyperplane contains a component of the curve.

D. Examples

  (i) disjoint union of a line and a conic

  (ii) disjoint union of a line and a plane curve. Note about degeneracy loci being all the same. (To see that two such curves with same degeneracy locus are linked, have to link directly. But note that modules are isomorphic anyway.)

(iii) Buchsbaum curves: no degeneracy loci

(iv) example on page 44 of thesis

(v) skew lines ($\mathbb{P}^3$ and $\mathbb{P}^4$), including case on a quadric. Note that the results in $\mathbb{P}^4$ are very similar to those in $\mathbb{P}^3$.

(vi) curves of low degree, especially rational sextic. (Degeneracy locus isn't enough for isomorphism.)

E. Lazarsfeld-Rao "e+4 $\Rightarrow$ recover C" result (generalize skew lines)

XV. Sufficient conditions in Codimension Two

A. Rao's main result (Math. Ann. paper)

B. Corollary: curves in $\mathbb{P}^3$

C. Questions

(i) Is there a module analog for Rao's theorem (codimension 2)?

(ii) What is the right theorem in higher codimension?

(iii) Can we even find an example, apart from aCM, of two schemes with the same modules but lying in different even liaison classes? (Conjecture: two skew lines in $\mathbb{P}^4$— recall results about liaison of lines in $\mathbb{P}^4$.)

(iv) Can we find an example of two surfaces in $\mathbb{P}^4$ which are in different even liaison classes, but for a general hyperplane the corresponding curves are evenly linked? Yes: use the surface from Decker-Ein-Schreyer which is quasi-Buchsbaum but the hyperplane section has module of type (1,1) but not Buchsbaum, and for the second surface maybe take a cone over the disjoint union of a line and a conic.

D. Examples

(i) double lines

(ii) $\mathcal{L}^h_{n_1,\dots,n_t}$

(a) Recall results about shifts

(iii) disjoint union of a line and a plane curve

XVI. Structure of an even liaison class in codimension two

A. Basic double links (special case of liaison addition)

B. How to get the deformations

(i) relation to Hilbert function and shift

(ii) Ballico-Bolondi results

C. Lazarsfeld-Rao, BM3 (no proof since these are now special case of BBM)

D. Ballico-Bolondi-Migliore

E. Results from BM4, BM5:

(i) things about maximal rank

(ii) do basic double links in increasing degree

(iii) partition of even liaison class (mention Ballico-Bolondi for structure of family)

(iv) single liaison addition

F. Martin-Deschamps-Perrin (curves in $\mathbb{P}^3$)

G. Bolondi (surfaces in $\mathbb{P}^4$)

XVII. Applications

A. Maggioni-Ragusa

B. Miró-Roig

C. stick figures (BM5)