MATH 652 – TOPICS IN ALGEBRA: ORDINARY REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE SPRING 2002

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Text: *Representations of Finite Groups of Lie Type*, François Digne and Jean Michel, LMS Student Texts **21**, 1991.

The complex semisimple Lie groups (and the compact Lie groups) have algebraic counterparts which make sense over any field k. When k is taken to be a finite field, these counterparts are finite groups. They are group theoretically quite significant: according to the classification theorem, they account for "most" of the finite simple groups. Moreover, their study is quite rich, as it typically involves an interaction between group theory and algebraic geometry (in postive characteristic).

An important example of one of the groups we have in mind is the group $\mathrm{SL}_2(\mathbf{F}_q)$ where $\mathbf{F}_q$ is the field of order $q = p^r$, p a prime number. This group consists of the $q(q-1)(q+1)$ matrices of size 2×2 with entries in $\mathbf{F}_q$ which have determinant 1.

The course will review the theory of ordinary representations of finite groups as in [6]. We will find that the characters of the irreducible complex representations of a finite group G form an orthonormal basis for the $\mathbf{C}$ vector space of all "class functions" on G. (The function $f : G \to \mathbf{C}$ is a class function if it is constant on the conjugacy classes of G). These irreducible characters encode important information about a group.

The course will also discuss the various groups of interest. Mostly, we shall content ourselves with descriptions of these groups as "classical" matrix groups, though we will try to describe the significance of the theory of semisimple algebraic groups (over an algebraic closure of a finite field) to our groups.

The question of describing irreducible characters in some useful way is typically quite difficult. Of course, any irreducible representation appears as a constituent of the "regular representation" $\mathbf{C}[G]$, but one does not learn much about e.g. the values of the irreducible characters in this way. More useful is the technique of *induction*: given a representation of a subgroup $H \leq G$, one can form the corresponding induced representation, and hope to find its irreducible constituents.

In the case of a finite group of Lie type, according to Harish-Chandra's "philosophy of cusp forms" one can account for many irreducible characters of G by induction from parabolic subgroups (though this accounting is "inductive"; the parabolic subgroups have as quotients smaller finite groups of Lie type whose irreducible representations are supposed by induction to be understood). The irreducible characters which can not be accounted for in this way are called *cuspidal*, and their construction is somewhat mysterious. In the case of $\mathrm{SL}_2(\mathbf{F}_q)$ and $p > 2$, the cuspidal representations are somehow associated with the "non-split" torus, i.e. the subgroup $S = \left\{ \begin{pmatrix} a & \alpha b \\ b & a \end{pmatrix} \mid a, b \in \mathbf{F}_q, a^2 - \alpha b^2 = 1 \right\}$ of order $q+1$, where $\alpha \in \mathbf{F}_q^\times$ is a fixed non-square.

In the case of the groups $\mathrm{GL}_n(\mathbf{F}_q)$, J. A. Green [4] worked out all of the irreducible representations in 1955. Bhama Srinivasan worked out the irreducible representations of the group $\mathrm{Sp}_4(\mathbf{F}_q)$. Deligne and Lusztig developed a general theory in [3], and we shall try to give a description of this theory.

The basic idea behind the construction of Deligne and Lusztig is the following: if G is the group of $\mathbf{F}_q$ rational points of an algebraic group, find an algebraic variety X defined over $\mathbf{F}_q$ on which the finite group of Lie type G acts. Then G acts on the ℓ-adic cohomology groups $H^*(X, \mathbf{Q}_\ell)$, and Grothendieck's trace formula permits one to describe the alternating sum of the characters of these G-representations.

Of course, we have no hope of describing in any detail the theory of ℓ-adic cohomology (!), but we will try to work out the case of $G = \mathrm{SL}_2(\mathbf{F}_q)$, where associated to the non-split torus S, one has a smooth algebraic curve C, of genus $\frac{q(q-1)}{2}$, on which G acts. The ℓ-adic cohomology of this curve may be describe in terms of the Tate group of its Jacobian, and the resulting representations of G are precisely the cuspidal ones.

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