George McNinch Math 651 – Topics in Algebra – Fall 1999 Linear Algebraic Groups

Text: *Linear Algebraic Groups*, 2nd edition, Tonny A. Springer, Progress In Mathematics Vol 9, Birkhäuser, 1998.

Overview: In this course, the objects of study are groups which have some geometric structure. A group which is also a smooth (real or complex) manifold (with certain natural stipulations that the group operations respect the geometry) is called a *Lie group*. In this course, we study objects analogous to Lie groups; the groups to be considered are *algebraic varieties* over an algebraically closed field, and are known as algebraic groups.

Algebraic groups play an important role in a number of different areas of mathematics. The much-studied "classical groups" (e.g. orthogonal, symplectic, unitary, general linear,... groups) are all examples of algebraic groups. From the point of view of finite group theory, algebraic groups are significant in that many of the now-classified finite simple groups arise from certain linear algebraic groups defined over fields of positive characteristic. Furthermore, there are numerous problems in geometry, number theory and algebra for which the study of algebraic groups has proved crucial.

The course will investigate the structure of algebraic groups. A connected linear algebraic group turns out to have a closed normal subgroup for which the quotient group is semisimple. (Actually, the normal subgroup can be taken connected and solvable as well.) Thus, to understand linear algebraic groups, one needs to study connected solvable groups and semisimple groups; the latter case turns out to be the most interesting. As in the study of compact Lie groups, or of semisimple complex Lie groups, there is a classification theorem for semisimple linear algebraic groups. (An algebraic group is linear when it is a closed subgroup of the group of all invertible $n \times n$ matrices for some $n > 0$; the notion of semisimplicity is a bit more complex to describe so I'll make you wait for the course!)

In fact, the classification of the three classes of groups just mentioned is basically the same: one associates to the group a combinatorial gadget called a *root system*, one proves that the root system more-or-less determines the group, and one classifies the possible root systems (indecomposable roots systems fall into 4 infinite families, with 5 exceptional diagrams. In particular, the classification of compact Lie groups, semisimple complex Lie groups, and semisimple algebraic groups is essentially the same).

The main goal of the course will be to work towards the classification theorem. There is quite a bit of preliminary work to do; fundamental results about algebraic groups must first be established. Once the preliminary results are obtained, we will begin work on the classification theorem. However, at this point in the course time constraints will make us prefer an emphasis on examples and explanation rather more than a complete proof which lacks examples.

If time permits, we will discuss (as does Springer's text) some aspects of the theory of linear algebraic groups defined over fields which are not algebraically closed.

Expectations:

A student who has completed the first year algebra course should be adequately prepared for the course. Some knowledge of basic algebraic geometry would be helpful, but is not required.

There will be no exams in the course. You are expected to attend class and to hand in solutions to the homework problems assigned periodically during the term.

Course Outline:

A nice feature of Springer's text is that it gives a self-contained account of the results from algebraic geometry and commutative algebra needed for the theory.

The following topics will be considered:

- Introduction to algebraic geometry.
- First properties of linear algebraic groups.
- Commutative algebraic groups.
- The Lie algebra of an algebraic group.
- Morphisms and quotients.
- Solvable groups, Parabolic subgroups.
- Weyl group, rank one groups.
- Reductive groups.
- classification issues
- (?) F-groups, F an arbitrary field.

Additional References:

(Caveat: No claim is made that this list is at all complete!!)

- General references for algebraic groups: [**?**] [**?**] [**?**] [**?**] [**?**] [**?**]
- Related Finite Groups: [**?**] [**?**] [**?**]
- Relations with Number Theory: [**?**] [**?**]
- Representation Theory [**?**]

References

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