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Isometric Embedding and the Monge-Ampère Equation

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Given a smooth n-dimensional Riemannian manifold (M^n, g), can we find a map $\phi : M^n \to R^N$ such that $\phi^\star h = g$, where h is the standard metric in R^N? Such a map provides a realization of the given Riemannian manifold in the Euclidean space R^N. This is a long-standing problem in Differential Geometry. We call the map ϕ an isometric embedding or isometric immersion if ϕ is an embedding or immersion. In the local version of the problem, one seeks the embedding in a neighborhood of any point.

The existence of global isometric embedding was first proved by Nash using a complicated iteration scheme in 1956. It was significantly simplified by Gunther in 1989 by using the contraction mapping principle. It is still an open problem what is the optimal dimension of the ambient Euclidean space.

The research on the local isometric embedding is focused on whether an n-dimensional Riemannian manifold (M^n, g) can be locally isometrically embedded in the Euclidean space R^{N_n} with $N_n = n(n+1)/2$. A simple inspection of the differential equation for the isometric embedding illustrates that N_n is the **right** dimension for the ambient space. The Cartan-Janet theorem, proved in 1930s, asserts that any analytic metric always admits a local analytic isometric embedding in R^{N_n}. Smooth case remains open, even for $n = 2$. For $n = 2$, the problem is reduced to the form whether an abstract surface can be locally isometrically embedded in R^3. In this case, the existence of the local isometric embedding is equivalent to solving a Monge-Ampère equation, whose type is determined solely by Gaussian curvature. In the general dimension, the influence of the curvature tensors is hard to recognize. So far, only partial results are known, based on a tedious discussion of *algebraic curves*. Understanding the type of the equation is only the initial step in solving the equation. Partial results on the existence of solutions, and hence the existence of the local isometric embedding, were obtained for $n = 2$ and $n = 3$. It should be emphasized that no counter example is known so far.

There is not a single book focused in this subject. Available results were presented in the original papers in different styles. Methods employed vary from differential equation, differential geometry to algebraic geometry. The aim of the course is to present these results in one framework. Basic knowledge on differential equation and differential geometry is required.