Facets of the Witten genus

Topics in Topology Course Fall 2002

Stephan Stolz

The Witten genus $\varphi_W(M)$ of an oriented manifold M is an element of $\mathbb{Q}[[q]]$ (i.e., a power series with rational coefficients). It should be thought of as a generalization of the more familiar $\widehat{A}$-genus $\widehat{A}(M) \in \mathbb{Q}$. The following table shows the various aspects of the $\widehat{A}$-genus and the φ_W-genus that we will cover in the course:

	$\hat{A}$-genus	Witten genus φ_W
assumption on M	M admits a spin structure	M admits a string structure
formal properties	$\widehat{A}(M)$ is an integer	Coefficients of $\phi_W(M)$ are integral; $\phi_W(M)$ is the 'q-expansion' of a modular form of weight $n/2$.
Topology	$M \mapsto \widehat{A}(M)$ induces a ring homomorphism $\widehat{A}\colon \Omega_*^{SO} \to \mathbb{Q}$; Ω_*^{SO} = bordism ring of oriented manifolds	$M \mapsto \varphi_W(M)$ induces a ring homomorphism $\varphi_W : \Omega_*^{SO} \to \mathbb{Q}[[q]]$
Analysis	$\widehat{A}(M)$ is the index of the Dirac operator on a manifold M	heuristically: $\varphi_W(M)$ is the S^1-equivariant index of the 'Dirac operator' on the free loop space $LM = \{\gamma\colon S^1 \to M\}$
Geometry	Theorem (Lichnerowicz): If M admits a Riemannian metric of positive scalar curvature, then $\widehat{A}(M) = 0$.	Conjecture (Höhn, Stolz): If M admits a Riemannian metric of positive Ricci curvature, then $\phi_W(M) = 0$.
Physics	$\widehat{A}(M)$ = partition function of a $0+1$-dimensional conformal field theory	heuristically: $\varphi_W(M)$ = partition function of a $1+1$-dimensional conformal field theory

Literature. The course will draw from many sources: the book by Hirzebruch-Berger-Jung [**?**] covers the topological aspects; for the analytical and geometrical aspects of the Dirac operator I recommend the book by Lawson-Michelsohn [**?**]. The interpretation of the Witten genus as S^1-equivariant index of the Dirac operator on loop spaces and its interpretation as a partition function is explained in the Witten papers [**?**],[**?**]. The conjectural relationship of the Witten genus and positive Ricci curvature is explained in my paper [**?**]. Later parts of the course will be based on recent joint work with Peter Teichner, presented in a series of fourteen lectures at the workshop 'topics on conformal field theory' in Münster, Germany in March of this year.

Prerequisites. The prerequisites for the course are the first year graduate courses in topology, algebra and analysis as well as a willingness to absorb a lot of definitions/

constructions/ theorems. To get something out out this course, participants should be willing to do some homework problems and to go over notes of previous lectures. The emphasis will be on developing concepts, doing examples and stating results rather than on presenting detailed proofs. The pace of the course will be somewhat rapid, but largely determined by the audience and suggestions from the audience.

Related courses/seminars. The course will continue in the spring semester, with focus on possible generalizations of the Witten genus for families of string manifolds parameterized by a space X; this 'families Witten genus' is an element of the elliptic cohomology of X. So far, there is only a *stable homotopy theoretic* construction of elliptic cohomology by Hopkins and Miller; we plan a seminar on that subject this fall with talks by the participants. Both, the course and the seminar have the same goal: gaining an understanding of the Witten genus; still, due to the differences of approach (geometric in the course, homotopy theoretic in the seminar), the direct points of contact will be somewhat spurious. Closely related to the conformal field theory point of view of the Witten genus is Katrina Barron's course this fall.

References

[HBJ] Hirzebruch, Friedrich; Berger, Thomas; Jung, Rainer, *Manifolds and modular forms*, Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, 1992. xii+211 pp.

[LM] Lawson, H. Blaine, Jr.; Michelsohn, Marie-Louise, *Spin geometry.* Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989. xii+427 pp.

[St] Stolz, Stephan, *A conjecture concerning positive Ricci curvature and the Witten genus*, Math. Ann. 304 (1996), no. 4, 785–800

[Wi1] Witten, Edward, *The index of the Dirac operator in loop space*, in 'Elliptic curves and modular forms in algebraic topology' (Princeton, NJ, 1986), 161–181, Lecture Notes in Math., 1326, Springer, Berlin, 1988

[Wi2] Witten, Edward, *Index of Dirac operators*, in 'Quantum fields and strings: a course for mathematicians', Vol. 1, 2 (Princeton, NJ, 1996/1997), 475–511, Amer. Math. Soc., Providence, RI, 1999