MATH 658
SPRING, 2002
W. Dwyer

This will be an advanced course in homotopy theory, in which the goal is to introduce some common methods and show how they are used in practice to solve problems. Some of the ideas to be discussed are

- The Serre spectral sequence, multiplicative structure, edge homomorphisms, transgression.
- The Steenrod algebra
- Transfer maps
- The Lefschetz fixed point theorem.

At least in the beginning, the focus will be on using these tools to study properties of compact Lie groups and of their classifying spaces. The outline for this will be as follows.

Finite groups. Show that if G is a finite group, then the mod p cohomology of the classifying space BG is finitely generated as an algebra. The technique is to embed G in the unitary group $U(n)$ and study the Serre spectral sequence of the fibration $U(n)/G \to BG \to BU(n)$. As part of the argument, it is necessary to calculate the cohomology of $BU(n)$.

Finite groups again. Start over. Show that if P is a finite p-group the mod p cohomology of BP is finitely generated as an algebra, by induction on the size of P. The argument involves looking at the Serre spectral sequence of a central extension of groups in which the kernel is Z/p, and using Steenrod operations to show that if the cohomology of the quotient group is finitely generated as an algebra, then the differential d_r is nontrivial for only a finite number of r. Then use transfer (to a Sylow p-subgroup) to show that if G is an arbitrary finite group the cohomology of BG is finitely generated as an algebra.

Compact Lie groups. Develop some counting principles relating Lefschetz numbers to Euler characteristics of fixed point sets. Use these to argue that inductively (on dimension) that a compact Lie group G contains a maximal torus, i.e. a toral subgroup T such that the Euler characteristic of G/T is nonzero. Show that the maximal torus is unique up to conjugacy. Identify the Weyl group as a group generated by reflections, and show that the rational cohomology of BG is given by the elements in the rational cohomology of BT which are invariant under the action of the Weyl group. Finally, use an inductive argument to show that the mod p cohomology of the classifying space of a p-toral group is finitely generated as an algebra, and then use transfer to conclude that the same is true of the cohomology of the classifying space of a general compact Lie group.

For this material the course will roughly follow

W. Dwyer and C. Wilkerson, *The elementary geometric structure of compact Lie groups*, Bull. London Math. Soc. 30(1998), 337–364.

In the second half of the semester, the course may continue along the same lines (with discussion of centers, product decompositions, etc.) or it may veer in a different direction, towards homotopy colimits and homology decompositions.